Can Algebraic Circuit Lower Bounds Have Easy Proofs?

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Part I

The Basics

Polynomials

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- Variables - $\bar{x} = \{x_1, \ldots, x_n\}$, constants - $\mathbb{F} = \mathbb{C}$
  Operations - Addition $+$ and multiplication $\times$.

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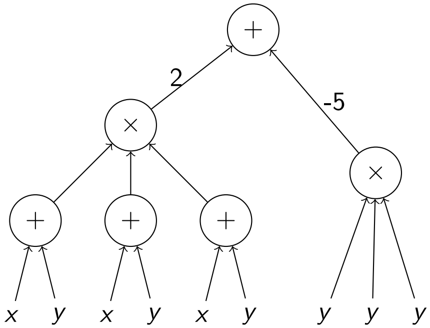

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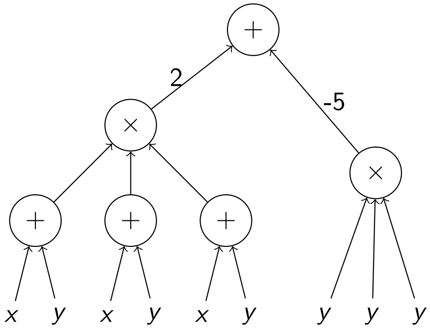

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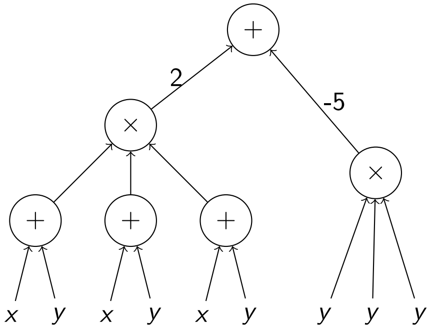

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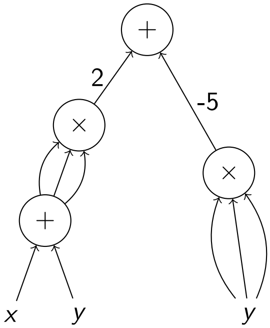

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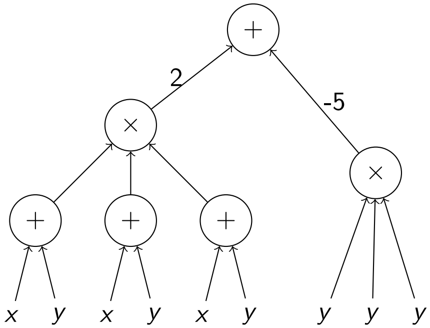

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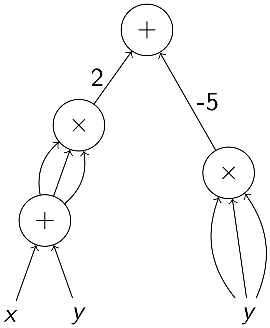

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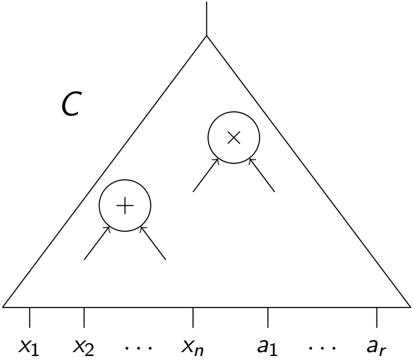
$f(\bar{x}) \in \mathbb{F}[\bar{x}]$



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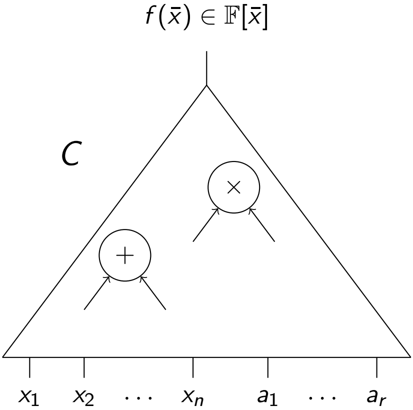

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Q. Can we give tight bounds on size(f) for a certain f?

size(f): Size of the smallest circuit computing $f(\bar{x})$.

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Q. VP vs VNP $\approx$ Det_n vs Perm_n.

Finding Hard Polynomials

- Best Lower Bounds:
  - Circuits: $\Theta(n \log d)$ for $x_1^d + \ldots + x_n^d$ [BS83,Smo97]
  - Formulas: $\Theta(n^2)$ for $\mathrm{ESym}(n, 0.1n)$ [CKSV20]

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- Progress in Restricted Models:
  - Constant depth circuits [NW95,KST16,GKKS13,...]
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Are current methods insufficient?

Part II

The Result

Proofs of Hardness/Non-membership

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  If $f(t) = at^2 + bt + c \in \mathcal{C}$, then $b^2 - 4ac = 0$.
- "Hard" Polynomial:
  $h(t) = a't^2 + b't + c'$ such that $b'^2 - 4a'c' \neq 0$.

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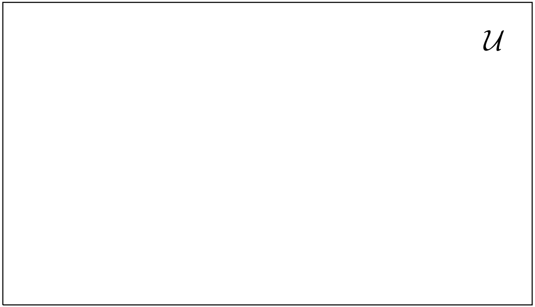

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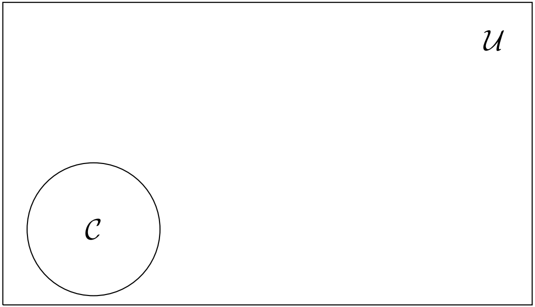

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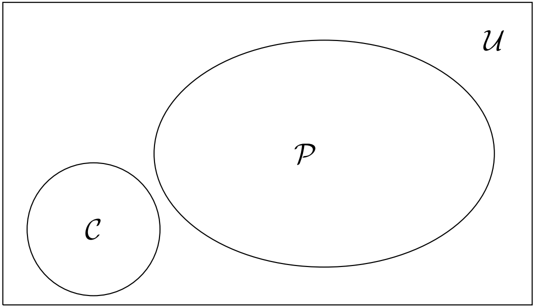

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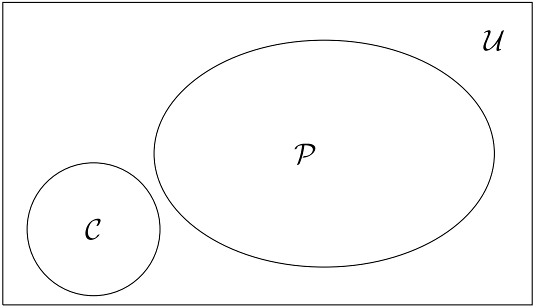

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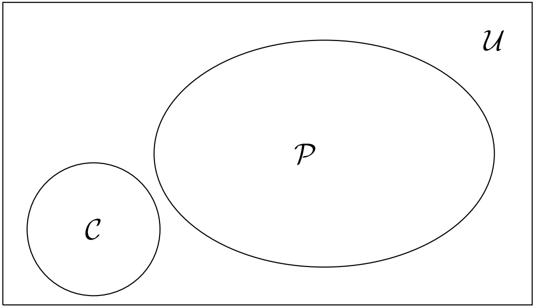

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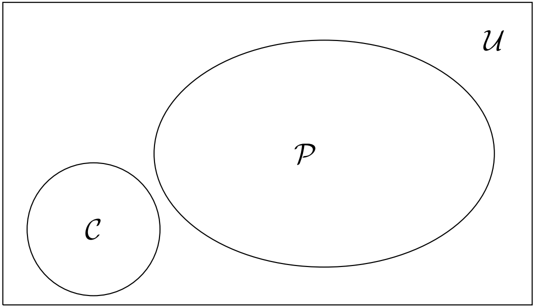

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- **Constructive**: "Easy" to decide if $f \in \mathcal{P}$, for all f.
- **Large**: "Most" $f \in \mathcal{P}$.

Lower Bounds from Natural Properties

$\mathcal{C}$ - class

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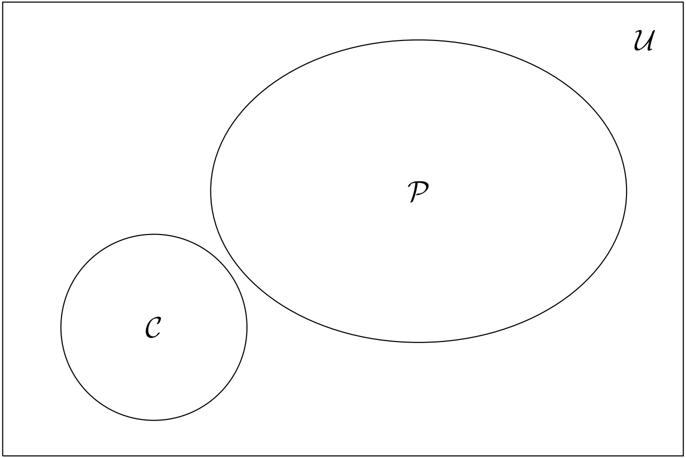

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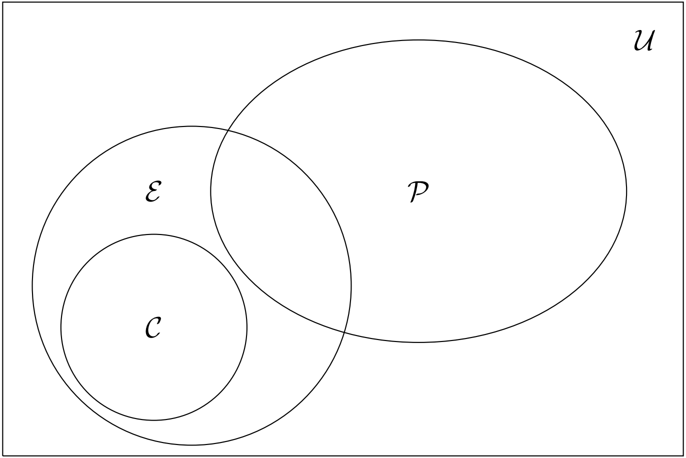

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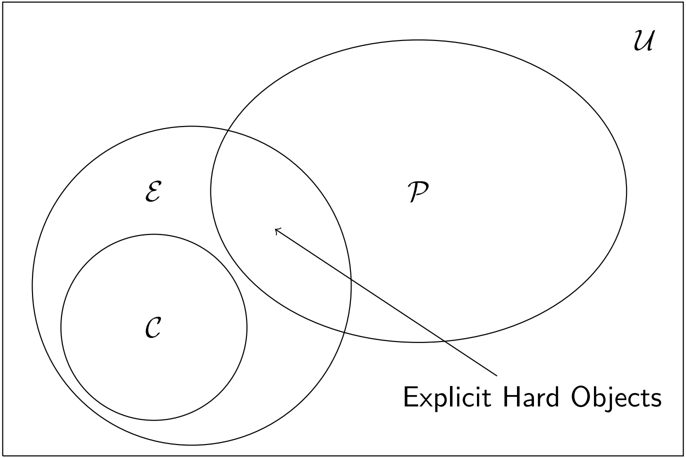

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$\mathcal{U} = \mathbb{F}^N$ $\mathcal{C} = \mathrm{VP}(n, d)$ $P_N =$ defining equation

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Q. Are there large VP_N-defining equations for VP_n?

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Q. What about natural proofs for VP?

Main Theorem

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Theorem [CKRST20] (Defining Equations for $\mathrm{VP}'_{\mathbb{C}}$):

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There exists a nonzero $P(Z_1, \ldots, Z_N) \in \mathrm{VP}(N)$ such that

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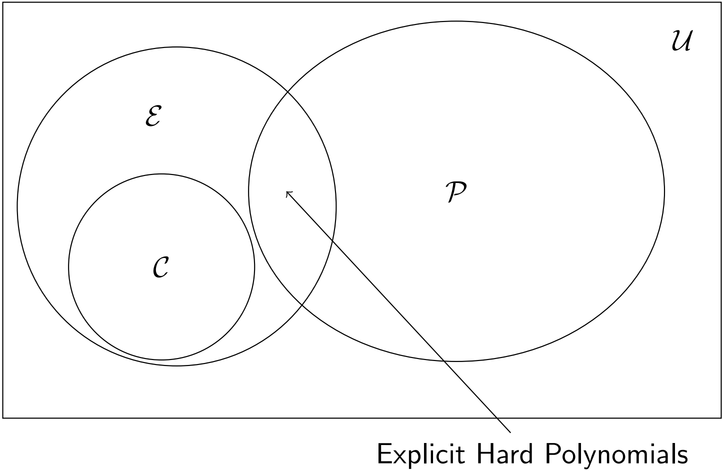
Q. How is this different from natural proofs?

Our Result and Natural Proofs

$\mathcal{C}$ - class, $\mathcal{P}$ - property, $\mathcal{E}$ - explicit polynomials



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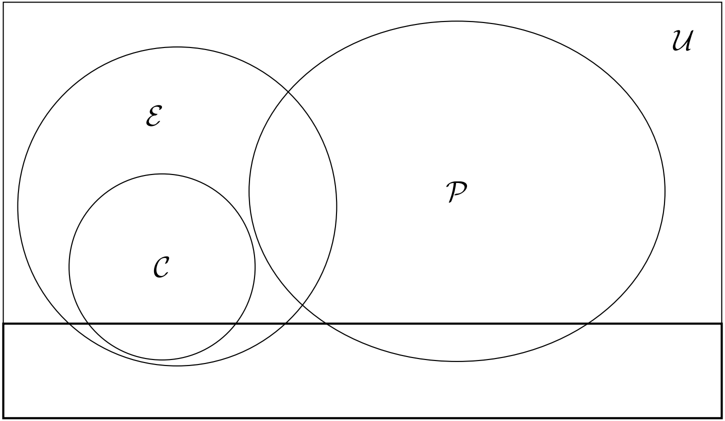

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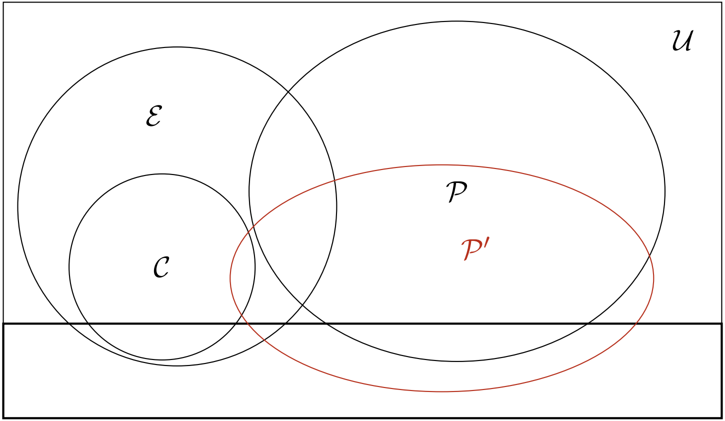

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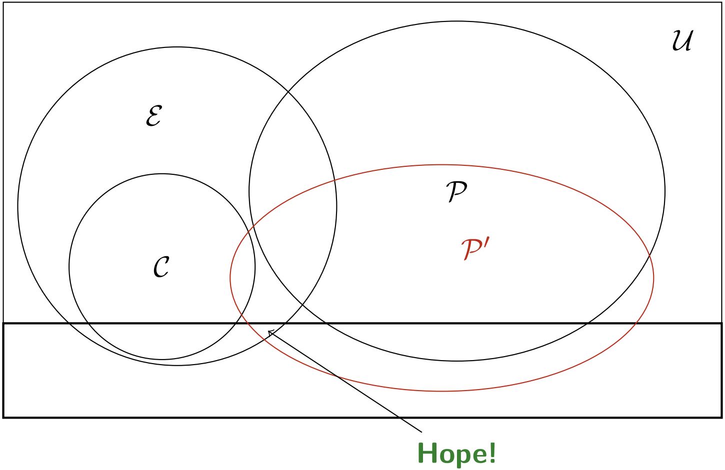

Part III

Proofs and Discussion

Defining Property of VP

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$\mathcal{H} \subset \mathbb{F}^n$ is a *hitting set* for a class $\mathcal{C}$ of n-variate polynomials, if for all $0 \neq f \in \mathcal{C}$, there exists an $h \in \mathcal{H}$ such that $f(h) \neq 0$.

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Hitting sets with small integers in the small coefficient setting.

Idea: For a nonzero g, $g(\mathcal{H}) = 0$ is a proof that $g \notin \mathcal{C}$.

Hitting Set as a Defining Property

$\mathcal{H} = \{h_1, \ldots, h_r\}$ hitting set for $\mathcal{C}$, $\quad 0 \neq g(\bar{x})$ input polynomial.

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$$
\begin{tikzcd}
\boxed{g(h_1)} & \boxed{g(h_2)} & \cdots & \boxed{g(h_{r-1})} & \boxed{g(h_r)} \\
 & & \text{coeffs}(g) \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
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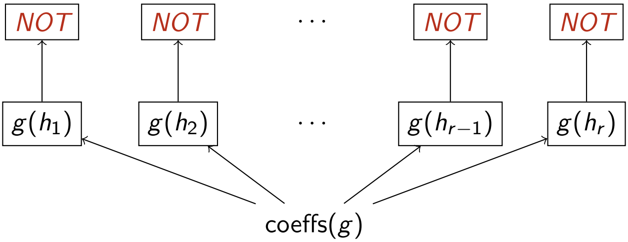

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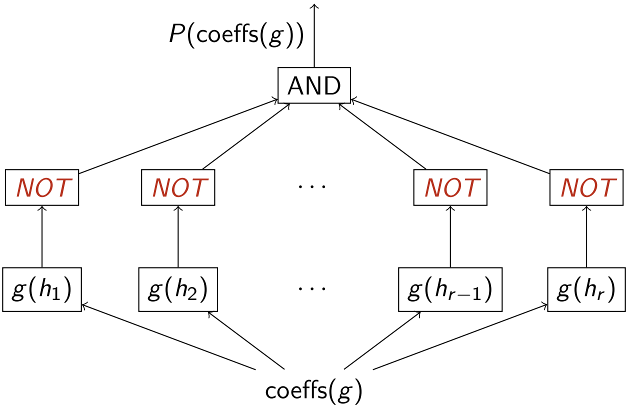

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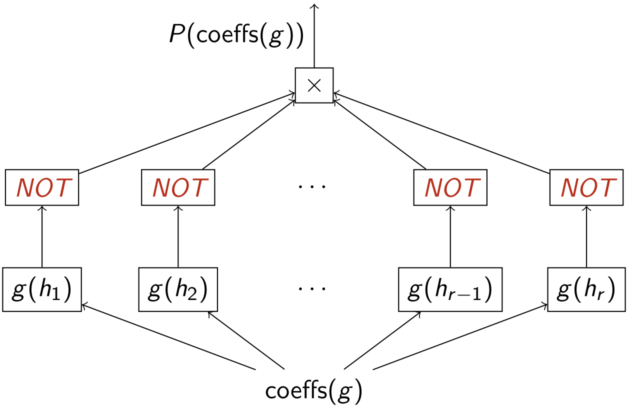

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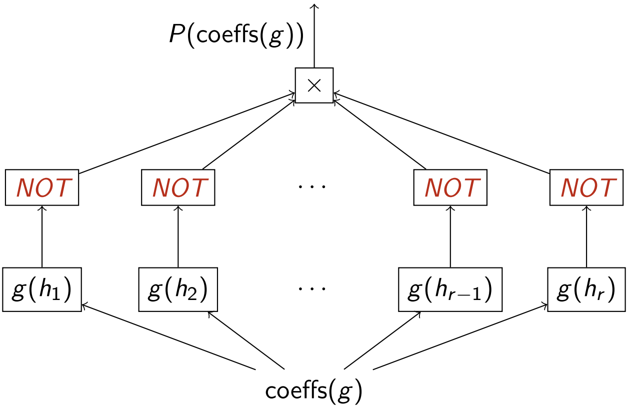

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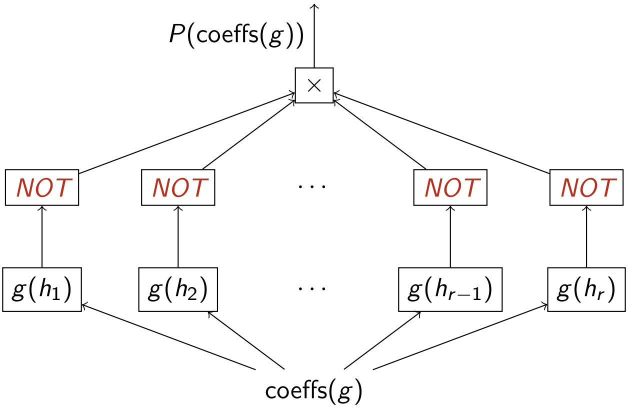

$NOT(0) =$nonzero $\quad NOT(\text{nonzero}) = 0$

Evaluating at a point

Given: Vector $\text{coeffs}(g_n) \in \mathbb{F}^N$, point $h \in \mathbb{F}^n$

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$$P(\text{coeffs}(g)) \approx \prod_{h \in \mathcal{H}} \left((\langle \text{coeffs}(g), \text{eval}(h) \rangle)^{q-1} - 1 \right)$$

$\text{Degree}(P) \leq |\mathcal{H}| q \leq \text{poly}(N)$, $\text{Size}(P) \leq \text{poly}(N)$.

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For $|\mathrm{coeffs}(g)| \leq L$, $|\langle \mathrm{coeffs}(g), \mathrm{eval}_i(h) \rangle| \leq \mathrm{poly}(N, L, d)$.

$$[g_n(h) = 0] \equiv \bigwedge_{i \in [\ell]} [\langle \mathrm{coeffs}(g), \mathrm{eval}_i(h) \rangle =_{p_i} 0]$$

Note: Can "hardwire" $\mathrm{eval}_i(h)$ for all $i \in [\ell]$ and $h \in \mathcal{H}$.

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$$[x =_{p_i} 0] \approx \prod_{\substack{-M \leq a \leq M \\ p_i \nmid a}} (x - a)$$

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Defining Equation for $\mathrm{VP}_{\mathbb{C}}$

$$P(\mathrm{coeffs}(g)) \approx \prod_{h \in \mathcal{H}} \prod_{i \in [\ell]} \prod_{\substack{-M \le a \le M \\ p_i \nmid a}} (\langle \mathrm{coeffs}(g), \mathrm{eval}_i(h) \rangle - a)$$

$\mathrm{Deg}(P) \le |\mathcal{H}|\, \mathrm{poly}(n)\, \mathrm{poly}(N) \le \mathrm{poly}(N)$

$\mathrm{Size}(P) \le \mathrm{poly}(N)$.

Results for VP

Theorem [CKRST20] (Defining Equations for $\mathrm{VP}'_{\mathbb{C}}$)

For n,d and $N = \binom{n+d}{n}$,

There exists a nonzero $P(Z_1, \ldots, Z_N) \in \mathrm{VP}(N)$ such that

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$P(\mathrm{coeffs}(f)) = 0$.

Theorem [CKRST20] (Defining Equations for $\mathrm{VP}_{\mathbb{F}}$)

For any fixed finite field $\mathbb{F}$, and n,d,$N = \binom{n+d}{n}$,

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Results for VNP

Theorem [CKRST20] (Defining Equations for $\mathrm{VNP}'_{\mathbb{C}}$)
For n,d and $N = \binom{n+d}{n}$,
There exists a nonzero $Q(Z_1, \ldots, Z_N) \in \mathrm{VP}(N)$ such that
for all $f \in \mathrm{VNP}_{\mathbb{C}}(n, d)$ with coefficients in $\{-N, \ldots, N\}$,
$Q(\mathrm{coeffs}(f)) = 0$.

Theorem [CKRST20] (Defining Equations for $\mathrm{VNP}_{\mathbb{F}}$)
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- Existence of defining equations cannot separate VP and VNP.

  The hitting sets and the defining equations are different.

About the Results

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- Non-explicitness of defining equations comes from the non-explicitness of the hitting sets used.

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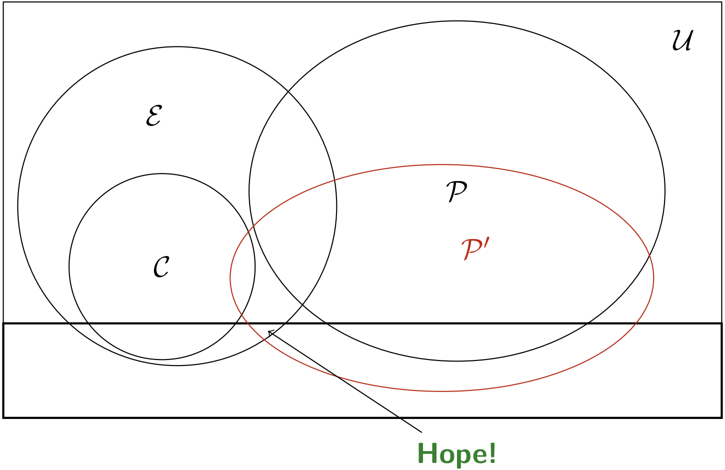
- Defining equations for other models not covered by [FSV18]?
  - Our proof technique will need small coefficients.



Thank You