The Brascamp-Lieb Inequality: Matroid Matching and Rank of Matrix Spaces

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Overview

- Brascamp-Lieb Theorem (with examples)
- Rank and Non-Commutative Rank of Matrix Spaces
- Fractional Linear Matroid Matching
- Algorithm and Proof
- Conclusion
• Brascamp-Lieb Theorem (with examples)

• Rank and Non-Commutative Rank of Matrix Spaces

• Fractional Linear Matroid Matching

• Algorithm and Proof

• Conclusion
Brascamp-Lieb Inequality

Given \( \{ B_j : \mathbb{R}^n \to \mathbb{R}^{n_j} \}_{j \in [m]} \) and \( c \in \mathbb{R}^m_+ \):

\[
\exists C < \infty? \quad \int_{\mathbb{R}^n} \prod_{j=1}^m f_j(B_j(x))^{c_j} \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^{n_j}} f_j(x_j) \right)^{c_j}.
\]
Brascamp-Lieb Inequality

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\]

Example: Holder’s inequality, Loomis-Whitney, Prekopa-Leindler, ...
Brascamp-Lieb Polytope

- Holder's inequality is true for all $c \in \mathbb{R}_+^m$ with $\sum_{j=1}^m c_j = 1$. 

Motivation: geodesic convex optimization, scaling framework, generalized submodular optimization

Pseudo-polynomial algorithms known, even $\text{NP/ coNP}$ open.
Brascamp-Lieb Polytope

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- In general, set of feasible exponents $x \in \mathbb{R}^m_+$ is a convex polytope:
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$$\forall V \subseteq \mathbb{R}^n : \sum_{j=1}^m x_j \cdot \dim(\text{im}(B_j^T) \cap V) \leq \dim(V)$$

$$\forall U_j \subseteq \text{im}(B_j^T) : \sum_{j=1}^m x_j \cdot \dim(U_j) \leq \dim(\sum_{j=1}^m U_j)$$
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Example: Linear Matroid Polytope

For \( \{ B_j := v_j^T \} \in [m] \), the polytope is linear matroid polytope

\[
P(V) := \{ x \in \mathbb{R}^m_+ : \forall S \subseteq [m] : x(S) \leq \text{rk}_V(S) \}.
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Definition

\( f : 2^[n] \rightarrow \mathbb{R} \) is submodular iff \( \forall S, T \subseteq [n] \):

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f(S) + f(T) \geq f(S \cup T) + f(S \cap T).
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Same definition holds for functions on lattice (with \( \vee, \wedge \)).
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Same definition holds for functions on lattice (with \( \lor, \land \)).

- \( S \to \text{rk}(V_S) \) is submodular on sets;
  - \( U \to -\dim(\text{im}(B^T) \cap U) \) is submodular on vector subspaces
  - \( \{U_j\} \to \dim(\sum_j U_j) \) is submodular on vector subspaces
• Brascamp-Lieb Theorem (with examples)

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Rank of a Matrix

\[ A \in \mathbb{F}^{m \times n}, \quad \text{rk}(A) = \ldots \quad (\text{audience participation}) \]
Rank of Matrix Space

Given $A_1, \ldots, A_k \subseteq \mathbb{R}^{m \times n}$ or $A := \langle A_1, \ldots, A_k \rangle$
Rank of Matrix Space

Given $A_1, ..., A_k \subseteq \mathbb{R}^{m \times n}$ or $\mathcal{A} := \langle A_1, ..., A_k \rangle$

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**Definition (Edmond's Problem)**

Given matrix space $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$, decide whether $\text{rk}(\mathcal{A}) = n$. 

**Theorem (Polynomial Identity Testing, KI04)**

Deterministic poly time algorithm for Edmond's Problem $\implies$ very strong arithmetic circuit lower bounds!
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- $= \text{rk}(A)$ for random $A \sim \mathcal{A}$ (algorithm)

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Given $A_1, \ldots, A_k \subseteq \mathbb{R}^{m \times n}$ or $\mathcal{A} := \langle A_1, \ldots, A_k \rangle$

- $\text{ncrk}(\mathcal{A}) := \text{rk}_{\mathbb{R}(\langle x \rangle)} \left( \sum_{i=1}^{k} x_i A_i \right)$

Theorem (GGOW15, IKQS15, HH21)

Non-commutative rank can be computed in deterministic poly time.
Non-Commutative Rank of Matrix Space

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- $\text{ncrk}(A) := \min \{ \dim(U) + \dim(V) \text{ s.t. } A|_{U,V} \equiv 0 \}$

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Properties

\[ \text{rk}(A) \leq \text{ncrk}(A) \leq 2\text{rk}(A) \]
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- \( \text{ncrk} \) is submodular minimization over vector spaces
Examples

- Bipartite graph:
Examples

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- Bipartite graph: $\max_{M \in \mathcal{M}} |M| = \text{rk} = \text{ncrk} = \min_{S \in \mathcal{VC}} |S|$

- General graph:
Examples

- Bipartite graph: $\max_{M \in \mathcal{M}} |M| = \text{rk} = \text{ncrk} = \min_{S \in \mathcal{V} \mathcal{C}} |S|$

- General graph: Digression: fractional matching polytope
Properties

- Invariant under change of basis $\mathbf{A} \rightarrow P\mathbf{A}Q$ for $\det(P) \det(Q) \neq 0$

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- Invariant under change of basis $A \rightarrow PAQ$ for $\det(P) \det(Q) \neq 0$

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Theorem (IKQS15)

$$\text{rk}(A) \leq \lim_{d \to \infty} \frac{\text{rk}(A^{\{d\}})}{d} = \max_{d \geq m, n} \frac{\text{rk}(A^{\{d\}})}{d} = \text{ncrk}(A)$$
Examples

- General graph:

\[ \text{Proposition (Lovasz, OS22)} \]

For \( \{ A_i, j := e_i e_T j - e_j e_T i \} \) \((i, j) \in G\),
\[ \max M \in M^2 \mid M \mid = \text{rk}(A) \neq n = \text{rk}(A) = \max x \in F M^2 \langle x, \vec{1} \rangle. \]

- Rank one inputs:

\[ \text{Proposition (Matroid Intersection)} \]

For \( \{ A_j := a_j b_T j \} \) \( j \in [m] \)
\[ \max S \in I A \cap I B \mid S \mid = \text{rk}(A) = n = \text{rk}(A) = \min S \subseteq [m] \text{rk}(A)(S) + \text{rk}(B)(S). \]
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For \( \{A_{i,j} := e_i e_j^T - e_j e_i^T\}(i,j) \in G\),

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• Brascamp-Lieb Theorem (with examples)

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Matroid Matching

**Definition (Lovasz)**

Given graph $G = (V, E)$ and matroid $\mathcal{M}$ on ground set $V$, find matching $M$ in $G$ such that $V(M)$ is independent in $\mathcal{M}$. 

In general: requires exponential $\mathcal{M}$ queries, also NP-hard.

**Theorem (Lovasz)**

Given pairs $\{(a_j, b_j) \subseteq F\}_{j \in [m]}$, let $A_j = a_j b_j^T - b_j a_j^T$. Then $\text{rk}(A)$ is twice the maximum linear matroid matching.
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Fractional Matroid Matching

**Definition (Van92)**

Given matroid $\mathcal{M} = (E, \mathcal{I})$ and a set of rank two lines $\{\ell_j \subseteq E\}_{j \in [m]}$, a fractional matroid matching $x \in \mathbb{R}^m_+$ satisfies

$$\forall F \in F(\mathcal{M}) : \sum_{j=1}^{m} x_j \cdot \text{rk}(\ell_j \cap F) \leq \text{rk}(F).$$
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- Fractional relaxation to matroid matching
- In general: can be optimized over in poly time given access to certain oracles [Chang et al, GP13]
- Linear matroids over $\mathbb{Q}$: exactly equivalent to BL polytope!
  But above algorithm leads to bit size explosion
Main Result

**Theorem (OS22)**

For pairs \( \{B_j := (a_j, b_j) \subseteq \mathbb{F}^n\}_{j \in [m]} \), then fractional matroid matching polytope is equivalent to the Brascamp-Lieb polytope on this input. Further, let \( \{A_j := a_j b_j^T - b_j a_j^T\} \). Then

\[
\text{ncrk}(A) = \max_{x \in \text{FMM}(B)} 2\langle x, \vec{1} \rangle = \max_{x \in \text{P}(B)} 2\langle x, \vec{1} \rangle.
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\]

**Corollary**

There is a **strongly poly** algorithm for rank 2 BL unweighted optimization.
• Brascamp-Lieb Theorem (with examples)

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For general weighted optimization over BL polytope, the optimal dual solution is supported on a chain

\[ V_1 \subseteq \ldots \subseteq V_k \subseteq \mathbb{R}^n. \]
Proposition (Chang et al)

For general weighted optimization over BL polytope, the optimal dual solution is supported on a chain

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Proposition (Chang et al)

For rank two Brascamp-Lieb input \( \{ B_j \}_{j \in [m]} \)

\[
\max_{x \in P(B)} 2\langle x, \vec{1} \rangle = \min_{(U \subseteq V) \text{ cover}} \dim(U) + \dim(V).
\]
Main Result

Proposition (OS22)

For Brascamp-Lieb input \{B_j = (a_j, b_j)\}_{j \in [m]}, let \{A_j := a_j \wedge b_j\}_{j \in [m]}.
Then

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\[
\text{ncrk}(\mathcal{A}) = \max_{x \in P(B)} 2\langle x, \vec{1} \rangle.
\]

Proof.

\((\leq)\): consider any cover \((U, V)\) of \(B\), then \(\mathcal{A}|_{\overline{U}, \overline{V}} \equiv 0\) so

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Main Result

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Then

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\]

Proof.

\( \leq \): consider any cover \((U, V)\) of \(\mathcal{B}\), then \(\mathcal{A}|_{U, V} \equiv 0\) so

\[
\text{ncrk}(\mathcal{A}) \leq \min_{(U \subseteq V) \text{ cover}} \dim(U) + \dim(V).
\]

\( \geq \): consider any \((U, V)\) such that \(\mathcal{A}|_{U, V} \equiv 0\), then \((U, V)\) is a cover so

\[
\text{ncrk}(\mathcal{A}) \geq \min_{(U \subseteq V) \text{ cover}} \dim(U) + \dim(V).
\]
Brascamp-Lieb Theorem (with examples)

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Conclusion
Main result [OS22] combined with ncrk algorithm [IKQS] gives strongly polynomial for unweighted optimization of rank 2 $BL$. How about weighted optimization? Even NP, coNP certificates are not known. Connections to other comb-opt questions? Connection to other notions of tensor rank?
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How about weighted optimization?

Even NP, coNP certificates are not known.
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- Connections to other comb-opt questions?
- Connection to other notions of tensor rank?
Garg, Gurvits, Oliveira, Wigderson (2015)
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