The Brascamp-Lieb Inequality: Matroid Matching and Rank of Matrix Spaces

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Overview

- Brascamp-Lieb Theorem (with examples)
- Rank and Non-Commutative Rank of Matrix Spaces
- Fractional Linear Matroid Matching
- Algorithm and Proof
- Conclusion

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Brascamp-Lieb Inequality

Given $\{B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}\}_{j \in [m]}$ and $c \in \mathbb{R}^m_+$:

$$\exists C < \infty ? \quad \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(B_j(x))^{c_j} \quad \leq \quad C \quad \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j(x_j) \right)^{c_j}.$$

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Example: Holder's inequality, Loomis-Whitney, Prekopa-Leindler, ...

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$$\forall V \subseteq \mathbb{R}^n : \sum_{j=1}^m x_j \cdot \dim(\mathrm{im}(B_j^T) \cap V) \leq \dim(V)$$

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Pseudo-polynomial algorithms known, even **NP/ coNP** open

Example: Linear Matroid Polytope

- For $\{B_j := v_j^T\}_{j \in [m]}$, the polytope is linear matroid polytope

$$P(V) := \{x \in \mathbb{R}_+^m : \forall S \subseteq [m] : x(S) \leq \mathrm{rk}_V(S)\}.$$

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Definition

$f : 2^{[n]} \to \mathbb{R}$ is submodular iff $\forall S, T \subseteq [n]$:

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- $S \to \mathrm{rk}(V_S)$ is submodular on sets;
  $U \to -\dim(\mathrm{im}(B^T) \cap U)$ is submodular on vector subspaces
  $\{U_j\} \to \dim(\sum_j U_j)$ is submodular on vector subspaces

Rank of a Matrix

$A \in \mathbb{F}^{m \times n}$, $\text{rk}(A) = \ldots$ (audience participation)

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Given matrix space $\mathcal{A} \subseteq \mathbb{R}^{n\times n}$, decide whether $\operatorname{rk}(\mathcal{A}) = n$.

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Theorem (Polynomial Identity Testing, KI04)

Deterministic poly time algorithm for Edmond's Problem
$\implies$ **very strong** *arithmetic circuit lower bounds!*

Non-Commutative Rank of Matrix Space

Given $A_1, \ldots, A_k \subseteq \mathbb{R}^{m \times n}$ or $\mathcal{A} := \langle A_1, \ldots, A_k \rangle$

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Theorem (GGOW15, IKQS15, HH21)

Non-commutative rank can be computed in deterministic poly time.

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- General graph: Digression: fractional matching polytope

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Theorem (IKQS15)

$$\mathrm{rk}(\mathcal{A}) \leq \lim_{d\to\infty} \frac{\mathrm{rk}(\mathcal{A}^{\{d\}})}{d} = \max_{d\geq m,n} \frac{\mathrm{rk}(\mathcal{A}^{\{d\}})}{d} = \mathrm{ncrk}(\mathcal{A})$$

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Proposition (Lovasz, OS22)

For $\{A_{i,j} := e_i e_j^T - e_j e_i^T\}_{(i,j)\in G}$,

$$\max_{M\in\mathcal{M}} 2|M| = \mathrm{rk}(\mathcal{A}) \neq \mathrm{ncrk}(\mathcal{A}) = \max_{x\in FM} 2\langle x, \vec{1}\rangle.$$

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Proposition (Matroid Intersection)

For $\{A_j := a_j b_j^T\}_{j\in[m]}$

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Given graph $G = (V, E)$ and matroid $\mathcal{M}$ on ground set V, find matching M in G such that $V(M)$ is independent in $\mathcal{M}$.

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Given pairs $\{(a_j, b_j) \subseteq \mathbb{F}^n\}_{j \in [m]}$, *let* $\{A_j := a_j b_j^T - b_j a_j^T\}$. *Then* $\mathrm{rk}(\mathcal{A})$ *is twice the maximum linear matroid matching.*

Fractional Matroid Matching

Definition (Van92)

Given matroid $\mathcal{M} = (E, \mathcal{I})$ and a set of rank two lines $\{\ell_j \subseteq E\}_{j \in [m]}$, a fractional matroid matching $x \in \mathbb{R}_+^m$ satisfies

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- Linear matroids over $\mathbb{Q}$: exactly equivalent to BL polytope!
  But above algorithm leads to bit size explosion

Main Result

Theorem (OS22)

For pairs $\{B_j := (a_j, b_j) \subseteq \mathbb{F}^n\}_{j\in[m]}$*, then fractional matroid matching polytope is equivalent to the Brascamp-Lieb polytope on this input. Further, let* $\{A_j := a_j b_j^T - b_j a_j^T\}$*. Then*

$$\mathrm{ncrk}(\mathcal{A}) = \max_{x \in FMM(B)} 2\langle x, \vec{1}\rangle = \max_{x \in P(B)} 2\langle x, \vec{1}\rangle.$$

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Corollary

There is a **strongly poly** *algorithm for rank 2 BL unweighted optimization.*

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Submodularity and BL Polytope Dual

Proposition (Chang et al)

For general weighted optimization over BL polytope, the optimal dual solution is supported on a chain

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Proposition (Chang et al)

For rank two Brascamp-Lieb input $\{B_j\}_{j\in[m]}$

$$\max_{x\in P(B)} 2\langle x, \vec{1}\rangle = \min_{(U\subseteq V)\ cover} \dim(U) + \dim(V).$$

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($\leq$): consider any cover (U, V) of B, then $\mathcal{A}|_{\overline{U},\overline{V}} \equiv 0$ so

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$(\geq)$: consider any (U, V) such that $\mathcal{A}|_{\overline{U},\overline{V}} \equiv 0$, then (U, V) is a cover so

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- How about weighted optimization?
- Even NP, coNP certificates are not known.
- Connections to other comb-opt questions?
- Connection to other notions of tensor rank?

References I

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