Combinatorial aspects of matrix multiplication

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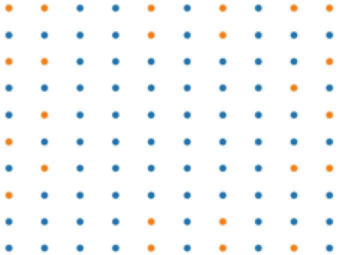

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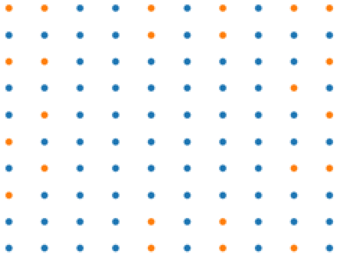

- **What does this have to do with algorithms for matrix multiplication?**

A billion dollar question

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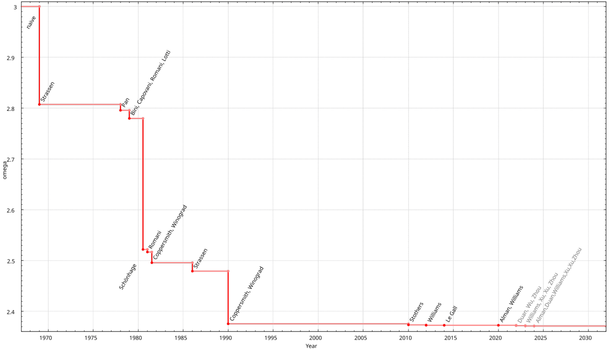

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- $\omega < 2.372$ [ADW+24]

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We know very little about the limits of current approaches!

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Theorem

Suppose X, Y, Z satisfy the TPP. Then $(|X||Y||Z|)^{\omega/3} \leq \sum d_i^{\omega}$, where d_i are the dimensions of the irreducible representations of G.

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Example

Let $G = \mathbb{Z}_m^3$.

$$X = (\mathbb{Z}_m, 0, 0), Y = (0, \mathbb{Z}_m, 0), Z = (0, 0, \mathbb{Z}_m)$$

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- Optimal for abelian groups: addition map $(x, y, z) \to x + y + z$ injective, so $|G| \geq n^3$

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- Intuition: if we can perform r independent instances of $n \times n$ matrix mult in time $|G|$, expect to perform 1 instance in time $|G|/r$

A nontrivial bound on ω

- In $\mathbb{Z}_m^3$, let

$$\begin{aligned} X_1 &= (*, 0, 0) & Y_1 &= (0, *, 0) & Z_1 &= (0, 0, *) \\ X_2 &= (0, *, 0) & Y_2 &= (0, 0, *) & Z_2 &= (*, 0, 0) \end{aligned}$$

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- Hence $\omega \leq \log_{m-1} m^3/2$. For $m = 16$, this gives $\omega \leq 2.816$

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Conjecture

For arbitrarily large n, there exists an abelian group G of order $n^{2+o(1)}$, and n pairs of subsets A_i, B_i with $|A_i||B_i| \geq n^{2-o(1)}$ satisfying the SDPP.

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Example (The virus)

Let $G = \mathbb{Z}_m^{2\ell}$. For $S \subset [2\ell], |S| = \ell$, define

$$A_S = \{x \in \mathbb{Z}_m^{2\ell} : x_i = 0 \text{ for } i \in S, x_i \neq 0 \text{ else}\}$$

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- Taking $m = 6, \ell \to \infty$, this gives $\omega < 2.48$

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For every m, there exists $c_m > 0$ such that no STPP in $\mathbb{Z}_m^n$ can yield a bound of $\omega \leq 2 + c_m$.

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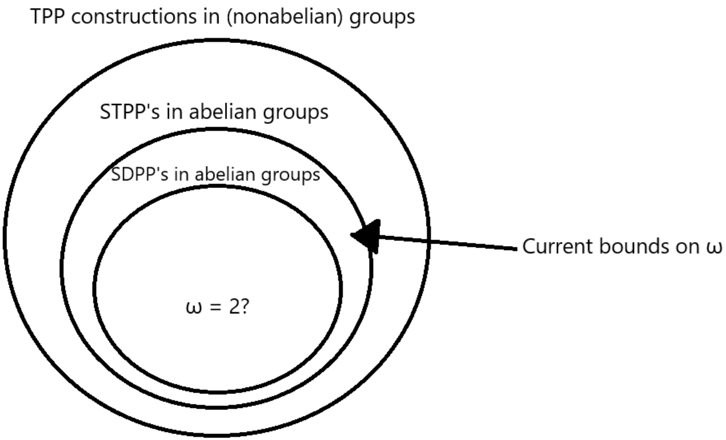
- $c_m \to 0$ as $m \to \infty$, so this says nothing for e.g. cyclic groups
- **Can we rule out STPP's in arbitrary abelian groups, and in particular, cyclic groups?**

Summary


TPP constructions in (nonabelian) groups
STPP's in abelian groups
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- STPP: induced disjoint union of M_n's in X_G

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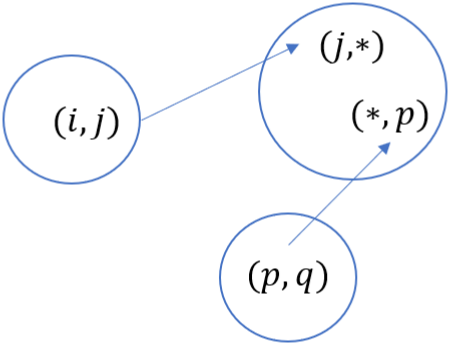

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Given a hypergraph X, let $Val(X)$ be the maximum number of hyperedges in any induced trapezoid-free subhypergraph of X.

- $Val(M_n) = n^3$ (maximum possible!)

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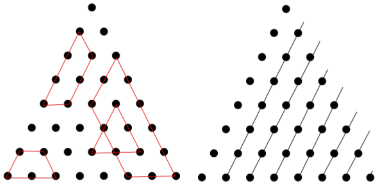
- Number of hyperedges = number of solutions to $x + y + z = 0$

Why “trapezoids”?

- Pictorially: let $\Delta_n := \{(x,y,z) \in [n] : x+y+z=n\}$



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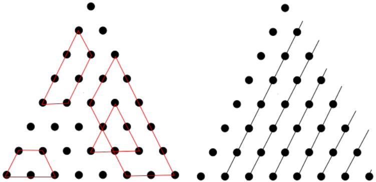

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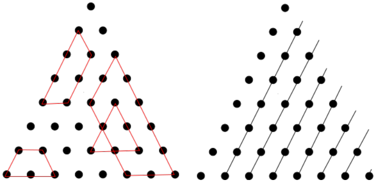

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- $Val(n)$: maximum number of points that can remain

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If $X \geq Y \implies Val(X) \geq Val(Y)$. So, $X_{\mathbb{Z}_n} \geq \sqcup_{i=1}^{k} M_N \implies Val(n) \geq kN^3$. □

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- Traditional corners: $(x, y), (x, y + d), (x + d, y)$
- Exist corner-free subsets of $[n]^2$ as big as $n^{2-o(1)}$ [AS74]

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Embed into $[(2m)^d]$ via base $2m$ expansion. □

Conclusion

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  - Would this imply that $\omega = 2$?
- Is “the virus” the best SDPP construction? Nikodym sets?

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Questions?

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