Fast Multivariate Multipoint Evaluation

Based on joint works with various subsets of

V Bhargava, S Ghosh, Z Guo, P Harsha, S Herdade, C K Mohapatra, R Saptharishi, C Umans

Multipoint evaluation

Input

- An m-variate polynomial f with degree at most (d-1) in each variable over a field **K**, as a list of coefficients
- N points $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbf{K}^{\mathbf{m}}$

Output

- Evaluation of f on $\alpha_1, \alpha_2, \ldots, \alpha_N$

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Input: $(d^m + Nm)$ field elements

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Naïve algorithm

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For i = 1 to N:

Evaluate f on α_i

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Can we do this faster ?

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Can we do this faster ?

In particular, is there an algorithm that runs in linear time in the input size ?

Multipoint evaluation <u>over infinite fields</u>

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Various versions: nearly linear time in the output size, input points with small absolute value, computing approximations of the evaluations, or count field operations only

Approximate multipoint evaluation (rationals/reals/complexes)

Input

- An m-variate polynomial f with degree at most (d-1) in each variable over rational numbers, as a list of coefficients
- N points $\alpha_1, \alpha_2, \ldots, \alpha_N \in Q^m$, from the unit cube
- Accuracy parameter t

Output

- Rational numbers $\beta_1, \beta_2, \ldots, \beta_N$ such that $|f(\alpha_i) - \beta_i| < 1/2^t$

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- Many direct and natural applications – fast modular composition, univariate polynomial factorization over finite fields, generating irreducible polynomials, computing minimal polynomials, data structures for polynomial evaluation, ...

- Current fastest algorithms for all these problems go via fast multipoint evaluation

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Multipoint evaluation: the univariate case

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Input is specified via $(N + d)$ field elements

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For structured set of input points

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- when $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbf{K}$ are all roots of unity of order N
- an algorithm with $(N + d)^{1+o(1)}$ field operations using Fast Fourier Transform

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- **[Borodin-Moenck, 1974]** An algorithm with $(N + d)^{1+o(1)}$ field operations

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- **[Borodin-Moenck, 1974]** An algorithm with $(N + d)^{1+o(1)}$ field operations
- a very clever and neat application of FFT

Multipoint evaluation: the univariate case

For an arbitrary set of input points

- **[Moroz, 2019]** An nearly linear time algorithm for approximate univariate MME
- Based on known algorithms for approximate FFT + beautiful geometric ideas

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- an easy nearly linear time algorithm – induction on the number of variables
- uses the univariate case as the base case

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- based on faster rectangular matrix multiplication

The multivariate case: more recent progress

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[Umans, 2008]

A nearly linear time algorithm for multivariate multipoint evaluation when

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A nearly linear time algorithm for multivariate multipoint evaluation when

1. **K** is any finite field
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Not a polynomial time algorithm, since the running time depends polynomially (and not polylogarithmically) on the field size

Nevertheless, happens to be very useful for one of our results

Multivariate multipoint evaluation

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This is the question that we study in our work and focus of rest of the talk.

Our results

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[Bhargava, Ghosh, K., Mohapatra, 2021]

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2. ~~number of variables (m) is less than $d^{o(1)}$~~

(degree d is asymptotically growing)

Our results

[Ghosh, Harsha, Herdade, K, Saptharishi, 2023]

A nearly linear time algorithm for approximate multivariate multipoint evaluation.

- Running time is $((Nm + d^m)t)^{1+o(1)}$

(degree d is asymptotically growing)

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No nearly linear time algorithm for multivariate multipoint evaluation when

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Nearly linear time algorithm for multivariate multipoint evaluation over all finite fields, for growing d, and all m

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Nearly linear time algorithm for approximate multivariate multipoint evaluation over rationals, reals, complex numbers, for growing d, and all m

In summary

	Field Size	Characteristic	Number of variables	Algebraic vs non-algebraic
Umans	Finite	char(K) < $d^{o(1)}$	m < $d^{o(1)}$	Algebraic
Kedlaya-Umans	Finite	All finite fields	m < $d^{o(1)}$	Non-algebraic
Bhargava-Ghosh-K-Mohapatra	Not-too-large	char(K) < $d^{o(1)}$	No constraint	Algebraic
Bhargava-Ghosh-Guo-K-Umans	Finite	All finite fields	No constraint	Non-algebraic
Ghosh-Harsha-Herdade-K-Saptharishi	Infinite	Rationals, Reals, Complex numbers	No constraint	Non-algebraic (approximate MME)

Applications ?

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- Two non-blackbox applications from the algorithm over finite fields of small characteristic – algebraic data structures for polynomial evaluation, upper bounds on the rigidity of Vandermonde matrices

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  - Space complexity: amount of memory needed for storage in the worst case
  - Query complexity: number of cells/bits in the memory needed to access queries in the worst case

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The *first* construction

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- Query complexity: $(d \cdot \log q)$ bits - not great

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A natural question

Is there a construction that achieves the best of both these worlds ?

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So, this *almost* answers this question!

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- algorithm of Kedlaya-Umans for multipoint evaluation is non-algebraic - uses things like bit operations, lifts the problem from the underlying field to over integers
- algebraic algorithms might be more aesthetic, could be useful when working with arithmetic models of computation

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If $q > \exp(d)$, then essentially no algebraic data structure for polynomial evaluation better than the trivial solution of storing all coefficients.

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[Miltersen, 1995]

If $q > \exp(d)$, then essentially no algebraic data structure for polynomial evaluation better than the trivial solution of storing all coefficients.

Conjectured that the same should hold even for smaller fields.

A corollary

[Bhargava, Ghosh, K., Mohapatra, 2021]

A data structure for polynomial evaluation with nearly linear space and sublinear query complexity, provided

1. char of the field is small
2. $q <$ quasipoly(d)

(Input size: $d \cdot \log q$ bits)

Matrix Rigidity

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(r,s)-rigid matrices

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[Valiant, 1977]

Explicit construction of sufficiently rigid matrices implies new lower bounds in algebraic complexity

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Many popular candidates – Hadamard, Discrete Fourier Transform, Vandermonde matrices

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No sufficiently strong lower bounds

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And, some more – Alman, Dvir-Edelman, Kivva….

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Rigidity upper bounds for general Vandermonde matrices remains open

A corollary

[Bhargava, Ghosh, K., Mohapatra, 2021]

Vandermonde matrices are not sufficiently rigid, when

1. char of the field is small
2. q < quasipoly(dimension)

An outline of the algorithm

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Theorem

A nearly linear time algorithm for multivariate multipoint evaluation when

1. char(**K**) is less than $d^{o(1)}$
2. **K** is of size at most exp(exp(exp(…exp(d)))) (tower of fixed height)

An outline of the algorithm

Input

- An m-variate polynomial f with degree at most (d-1) in each variable over a field ***K***, as a list of coefficients
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Two phases of the algorithm

- Preprocessing phase: independent of the evaluation points $\alpha_1, \alpha_2, \ldots, \alpha_N$

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Two phases of the algorithm

- Preprocessing phase: independent of the evaluation points $\alpha_1, \alpha_2, \ldots, \alpha_N$
- Local computation phase: depend on $\alpha_1, \alpha_2, \ldots, \alpha_N$, and earlier computation

An outline of the algorithm

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2. Evaluate f on all points of S

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Local computation

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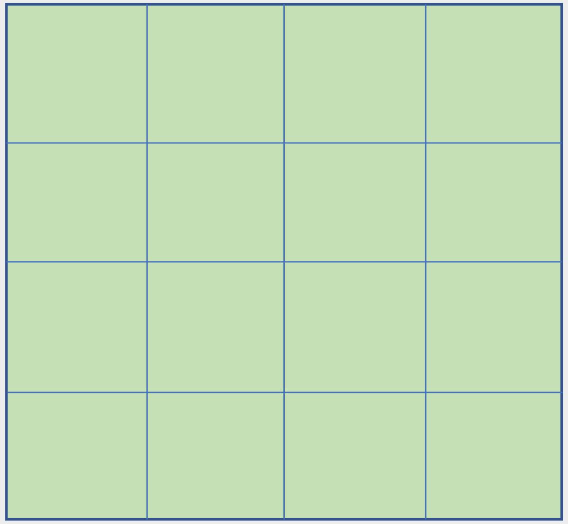
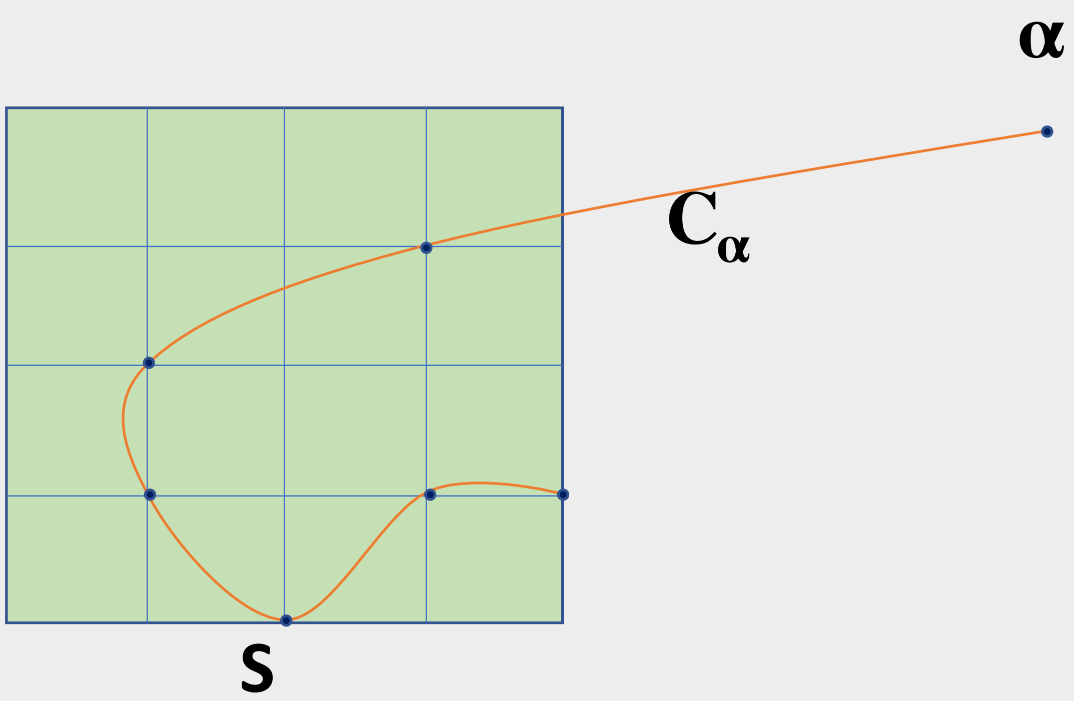
An outline of the algorithm

α



S

An outline of the algorithm



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- if we can efficiently get our hands on g, we can set t = u, to get $f(\alpha)$

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- from the preprocessing phase, we have already computed $\mathrm{g}(\mathrm{v})=\mathrm{f}(\mathrm{r}_{\alpha,1}(\mathrm{v}), \mathrm{r}_{\alpha,2}(\mathrm{v}), \ldots, \mathrm{r}_{\alpha,\mathrm{m}}(\mathrm{v}))$

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- once, we have g, can recover $g(u) = \ f(\alpha)$

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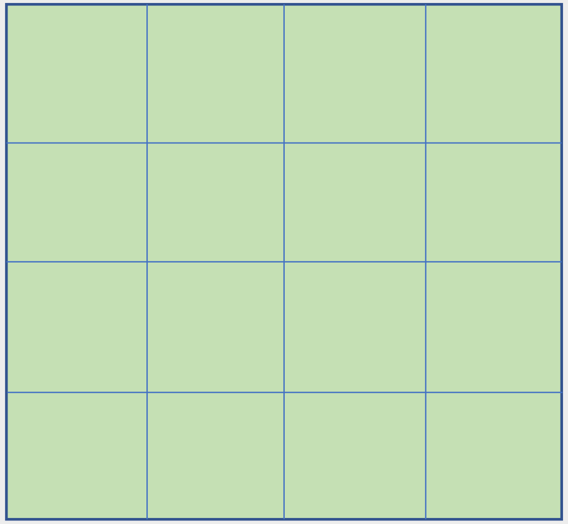

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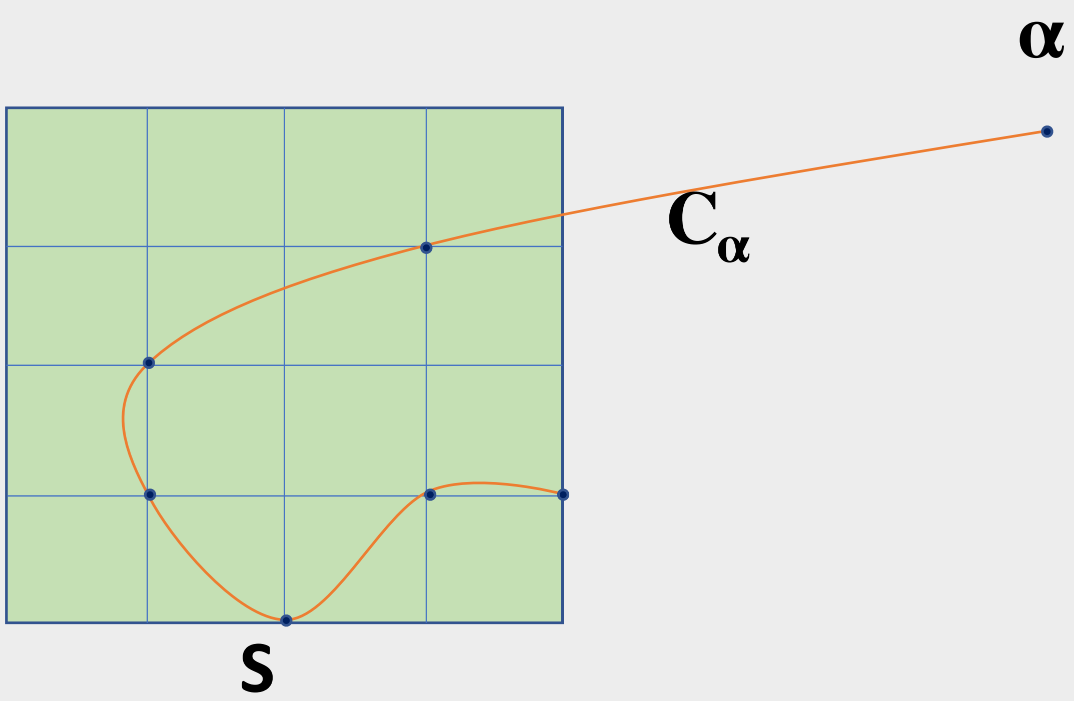

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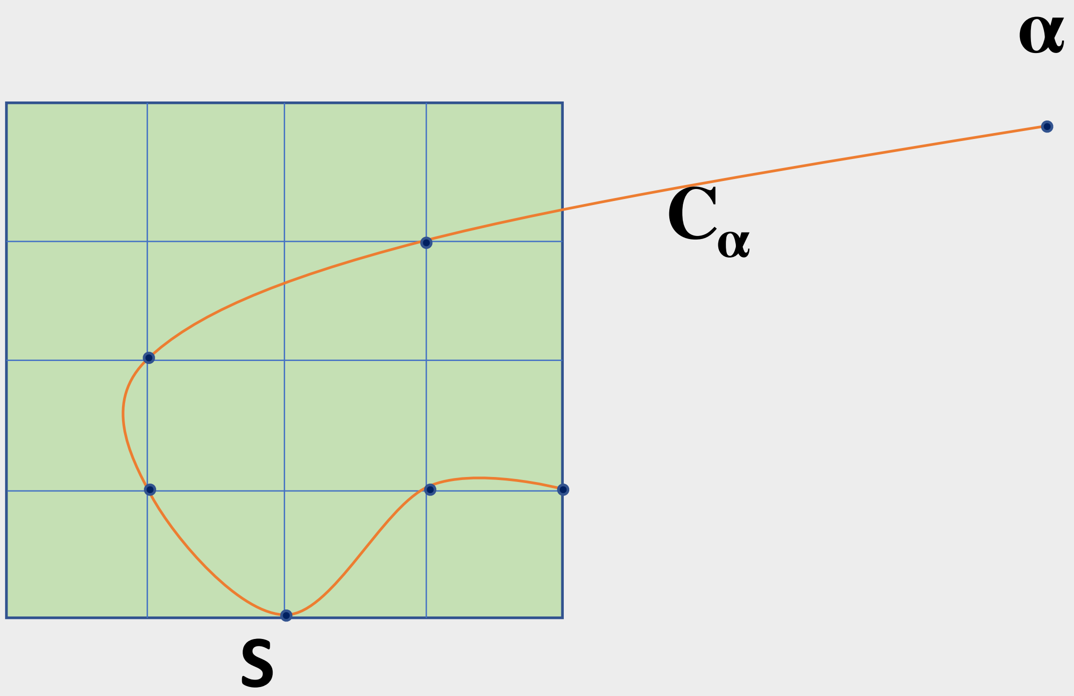

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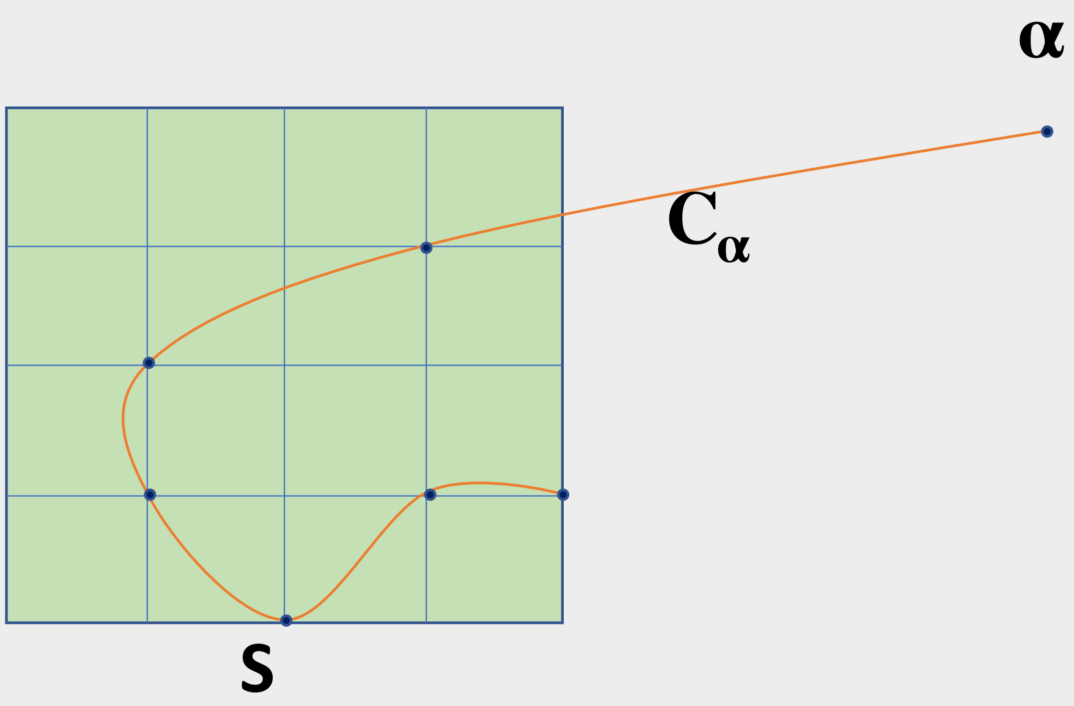

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- $|S| < \left(pdm \cdot \log_p |\mathbf{K}|\right)^m$
- $\deg(C_\alpha) < \log_p |\mathbf{K}|$
- $|C_\alpha \cap S| > \log_p |\mathbf{K}| \cdot dm > \deg(C_\alpha)\cdot dm$

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- has the mysterious curve property needed for subsequent step

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- to compensate, need stronger preprocessing phase, and a more complicated local computation step

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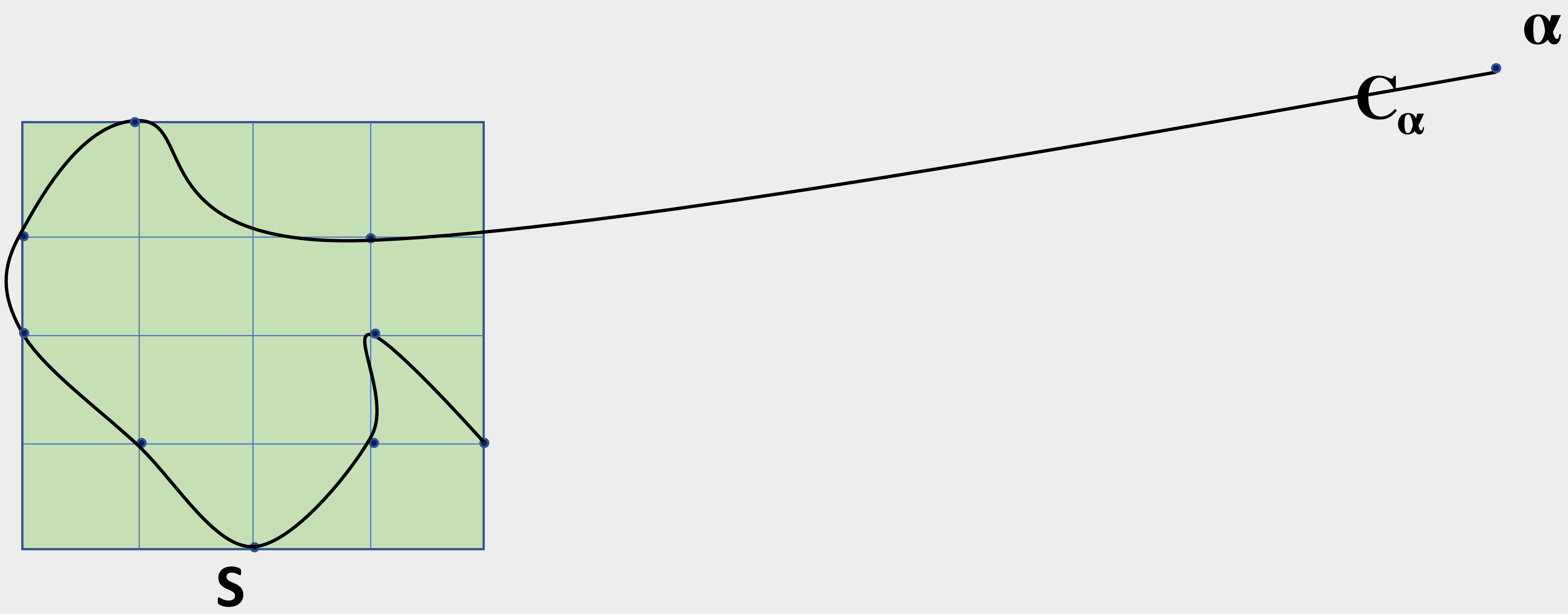
- then, use this additional info, together with values of f on S to do interpolation

The final inaccurate picture



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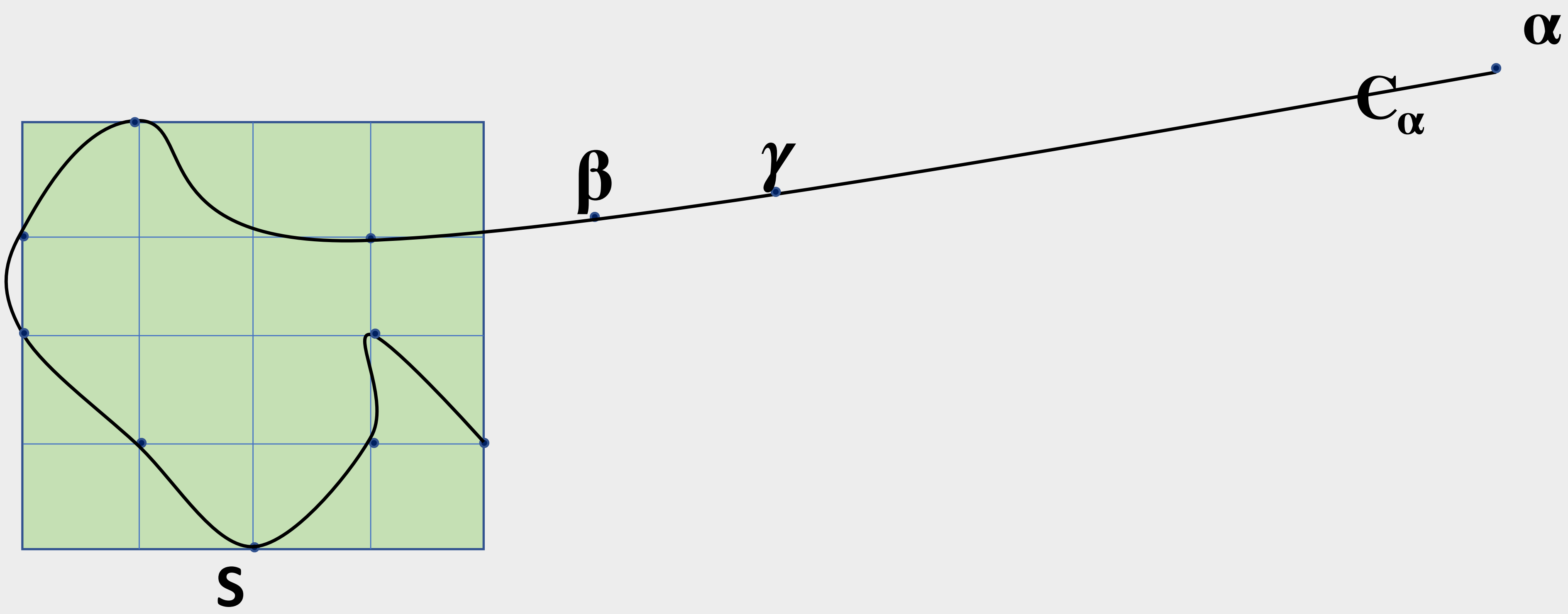

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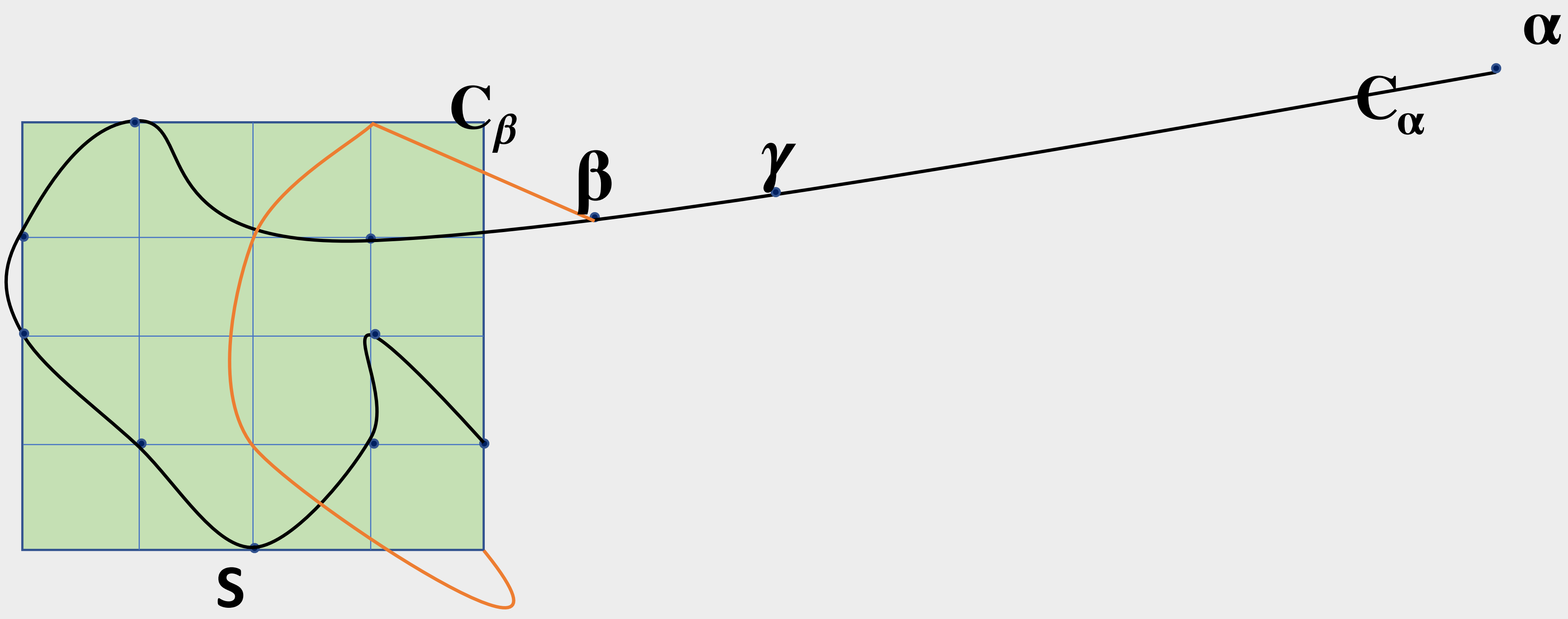

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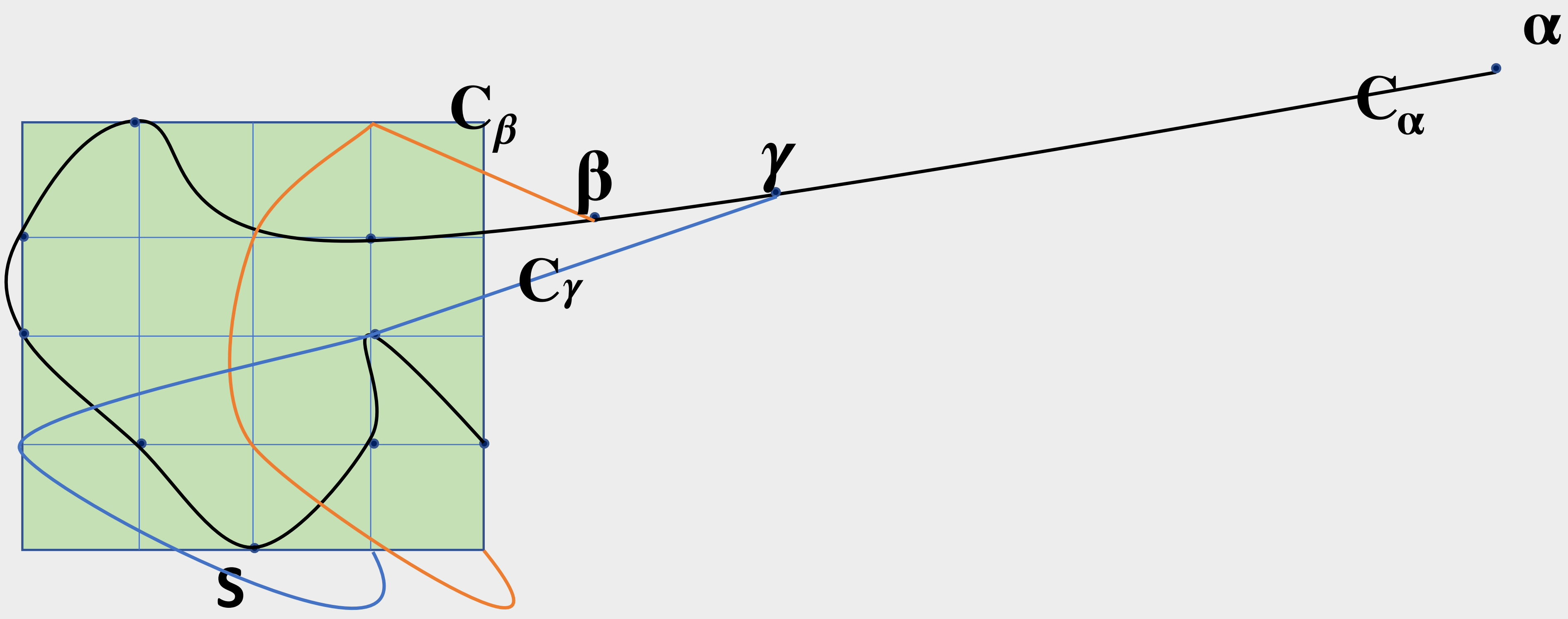

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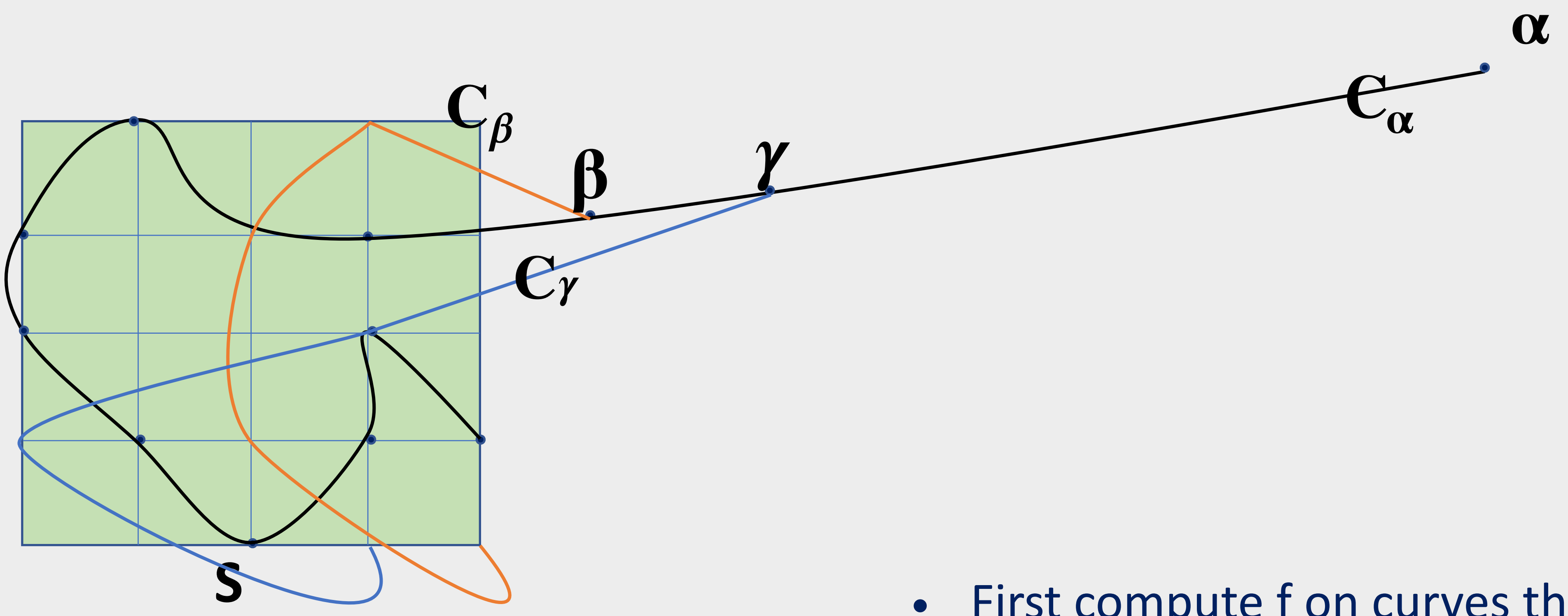

- First compute f on curves through simpler points $\beta,\ \gamma$ using the previous algorithm
- Then, use the values of f on S, and curves through $\beta,\ \gamma$ to compute f on C_α

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- crucially uses a result of Bombieri-Vinogradov about the density of primes in an arithmetic progression
- essentially, both improve some of the bottlenecks in Kedlaya-Umans using ideas from the small characteristic case and BKW19 in slightly different ways

Open Questions

- An algebraic algorithm over finite fields ?
- An algorithm (or an algebraic circuit) over infinite fields (complex numbers) ?
- More applications ?
- What about faster algorithms for other related problems ? e.g. multivariate interpolation ?
- What about the case of constant d ? e.g. multilinear polynomials ?

Thank You!