AN OPTIMAL SEPARATION OF RANDOMIZED AND QUANTUM QUERY COMPLEXITY

Alexander Sherstov, Andrey Storozhenko, and Pei Wu

UCLA

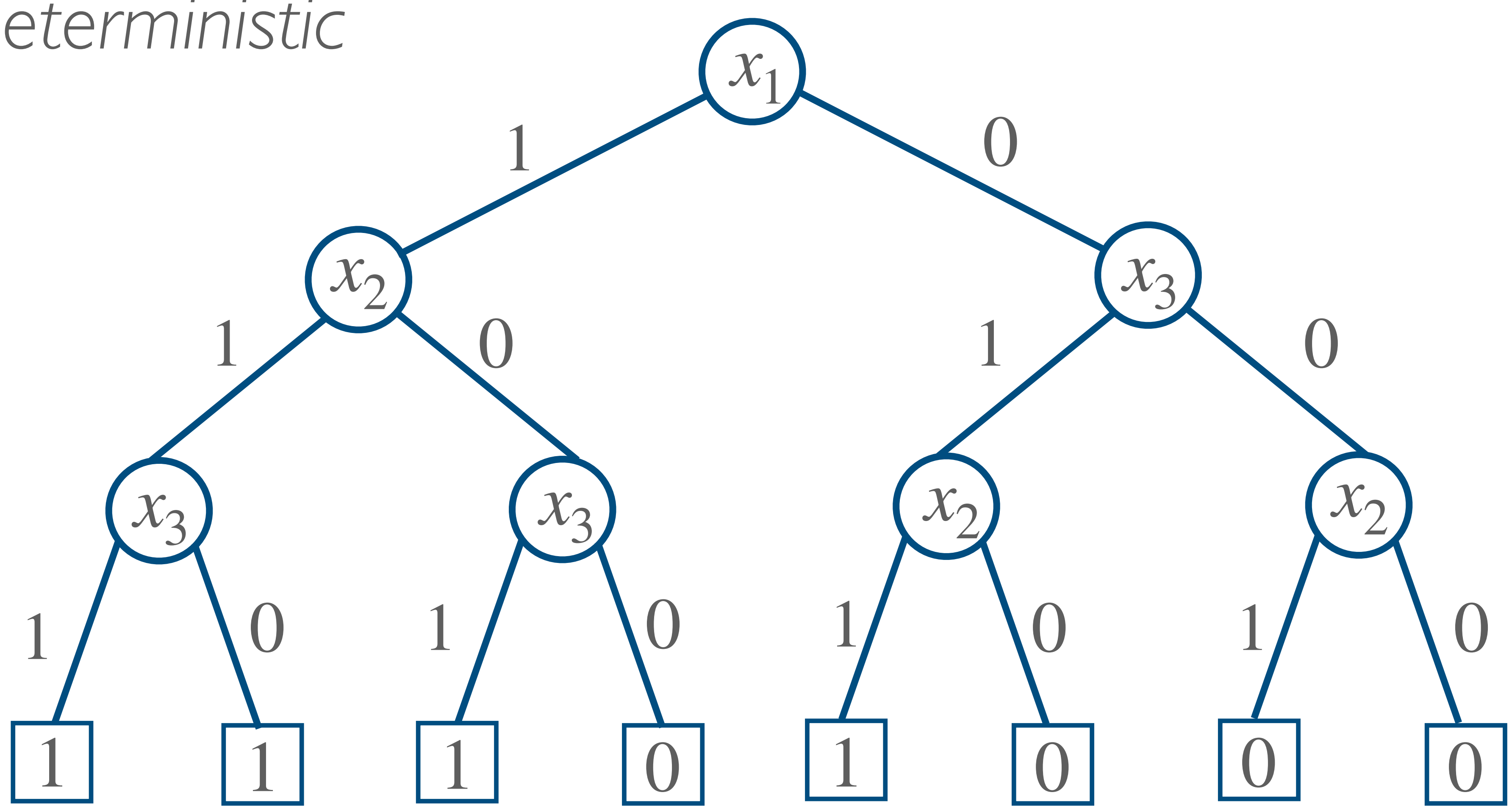
Central open problem

How much faster can quantum computers be than classical?

Most research focuses on the query model.

Query complexity

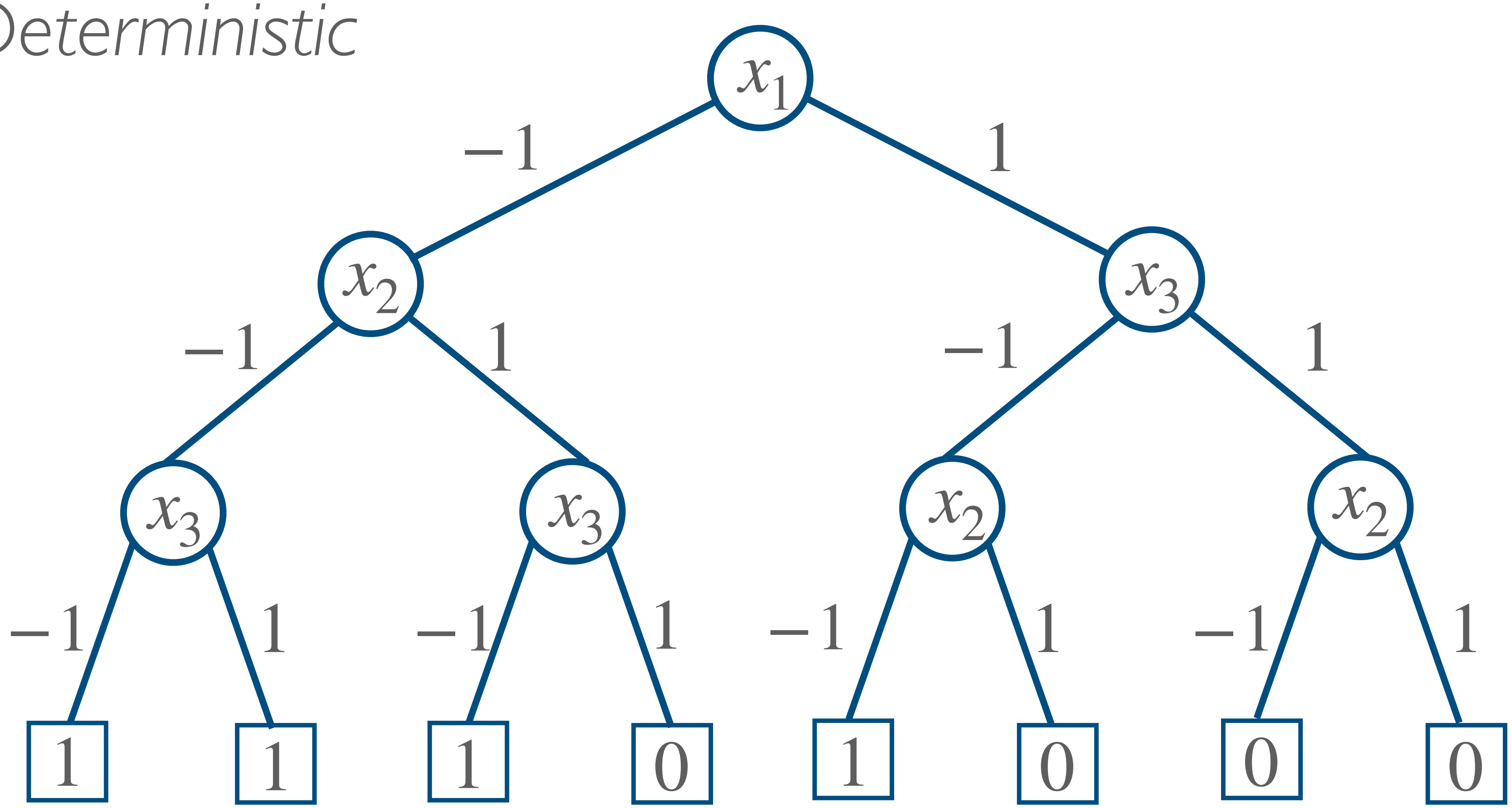
Deterministic



$$T : \{0,1\}^n \to \{0,1\}$$

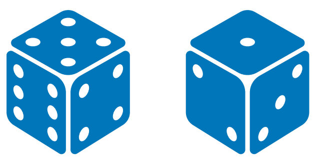
Query complexity

Deterministic

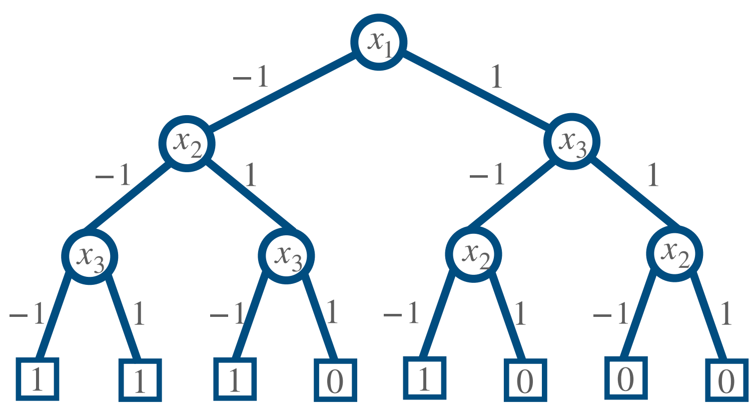


$$T : \{-1,1\}^n \rightarrow \{0,1\}$$

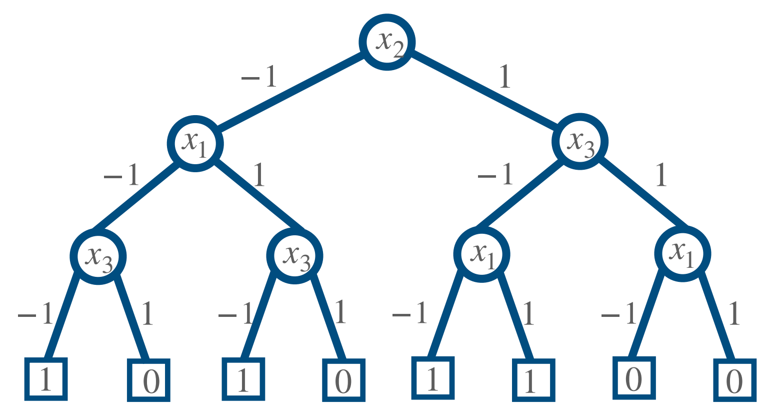
Query complexity



Randomized



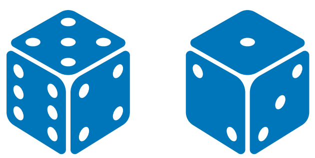
T_1



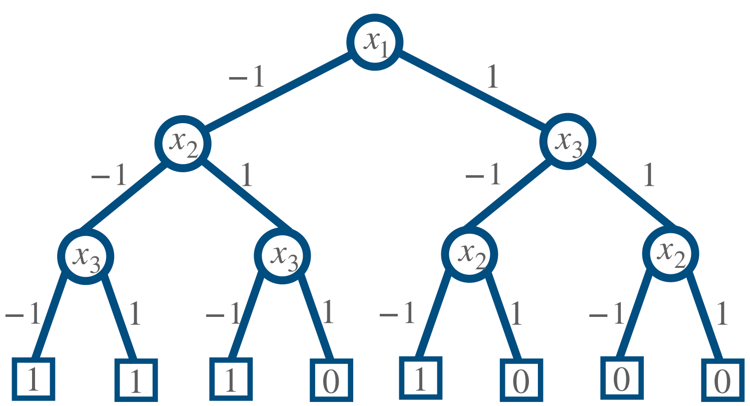
T_2

$\cdots$

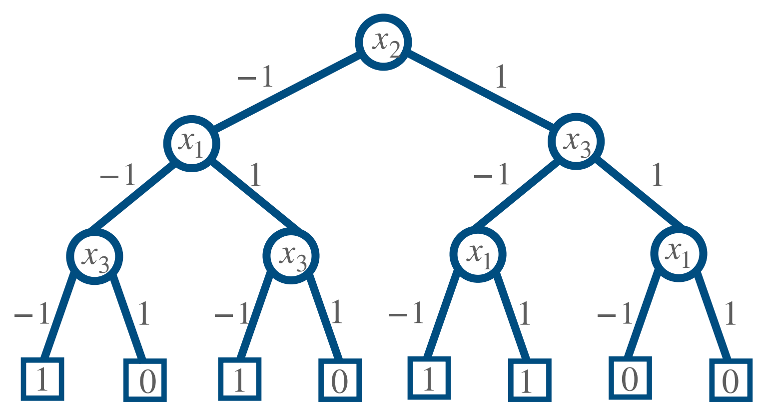
Query complexity



Randomized



T_1



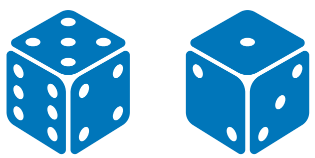
T_2

$\cdots$

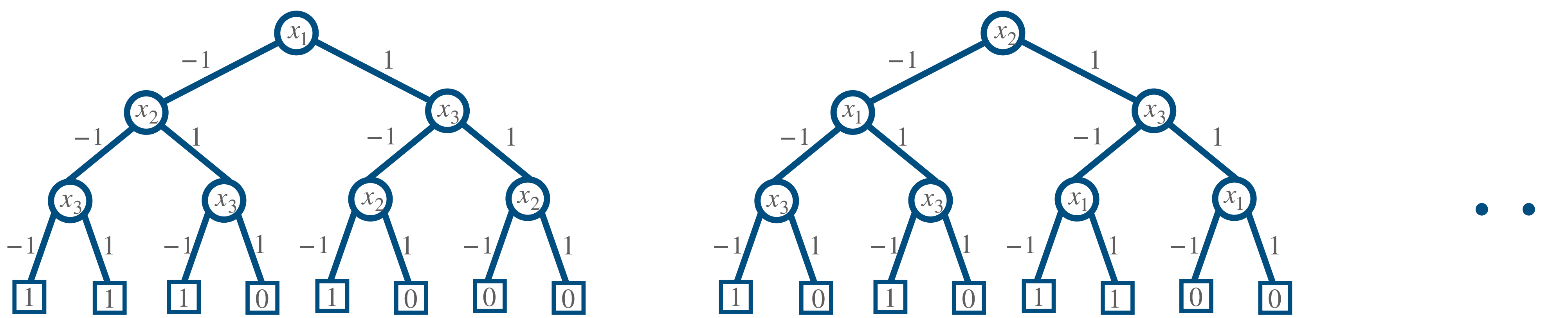
T computes $f : \{-1,1\}^n \to \{0,1\}$ with error ϵ if

$$\mathbf{P}_r[T_r(x) \neq f(x)] \leq \epsilon, \qquad \forall x \in \{-1,1\}^n.$$

Query complexity



Randomized

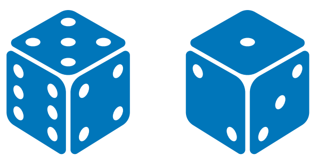


T_1 T_2

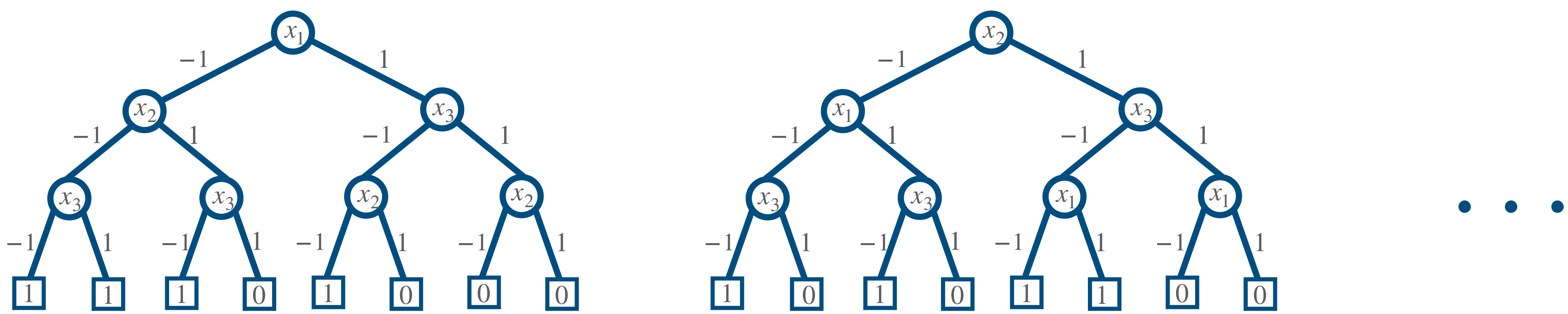
T computes $f : \{-1,1\}^n \to \{0,1,*\}$ with error ϵ if

$$\mathbf{P}_r[T_r(x) \neq f(x)] \leq \epsilon, \qquad \forall x \in f^{-1}(0) \cup f^{-1}(1)\,.$$

Query complexity



Randomized



T_1 T_2

$R_\epsilon(f)$ = minimum depth of a randomized decision tree for f with error ϵ.

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

query
index

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

workspace

query
index

Quantum query complexity

Quantum query

workspace

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

query
index

$$\downarrow$$

$$|\phi'\rangle = \sum_{i,w} a_{i,w} x_i |i\rangle |w\rangle$$

Quantum query complexity

Quantum query

workspace

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

query index

$$\downarrow$$

$$|\phi'\rangle = \sum_{i,w} a_{i,w} x_i |i\rangle |w\rangle$$

can access all x_i in a single query!

Quantum speedups

Query model captures nearly all quantum breakthroughs:

Deutsch-Jozsa's algorithm

Simon's algorithm

Grover's search

Bernstein-Vazirani's algorithm

Shor's factoring algorithm

……

Quantum speedups

Reference	Randomized	Quantum
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	Randomized	Quantum
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	Randomized	Quantum
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$
Aaronson-Ambainis 15	$\tilde{\Omega}(\sqrt{n})$	1

"forrelation"

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	Randomized	Quantum	
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$	
Aaronson-Ambainis 15	$\tilde{\Omega}(\sqrt{n})$	1	"forrelation"
	$O_k(n^{1-\frac{1}{k}})$	$k/2$	simulation

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	**Randomized**	**Quantum**	
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$	
Aaronson-Ambainis 15	$\tilde{\Omega}(\sqrt{n})$	1	"forrelation"
	$O_k(n^{1-\frac{1}{k}})$	$k/2$	simulation

$R(f) = \Omega(n), Q(f) = O(1)$

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	Randomized	Quantum	
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$	
Aaronson-Ambainis 15	$\tilde{\Omega}(\sqrt{n})$	1	"forrelation"
	$O_k(n^{1-\frac{1}{k}})$	$k/2$	simulation

$\cancel{R(f) = \Omega(n), Q(f) = O(1)}$

Impossible!

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	Randomized	Quantum	
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$	
Aaronson-Ambainis 15	$\tilde{\Omega}(\sqrt{n})$	1	“forrelation”
Aaronson-Ambainis 15	$O_k(n^{1-\frac{1}{k}})$	$k/2$	simulation
Tal 19	$\tilde{\Omega}(n^{\frac{2k-2}{3k-1}})$	$k/2$	“rorrelation”

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	Randomized	Quantum	
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$	
Aaronson-Ambainis 15	$\tilde{\Omega}(\sqrt{n})$	1	"forrelation"
Aaronson-Ambainis 15	$O_k(n^{1-\frac{1}{k}})$	$k/2$	simulation
Tal 19	$\tilde{\Omega}(n^{\frac{2k-2}{3k-1}})$	$k/2$	"rorrelation"
Our work	$\tilde{\Omega}(n^{1-\frac{1}{k}})$	$k/2$	"rorrelation"

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

Reference	Randomized	Quantum	
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$	
Aaronson-Ambainis 15	$\tilde{\Omega}(\sqrt{n})$	1	"forrelation"
Aaronson-Ambainis 15	$O_k(n^{1-\frac{1}{k}})$	$k/2$	simulation
Tal 19	$\tilde{\Omega}(n^{\frac{2k-2}{3k-1}})$	$k/2$	"rorrelation"
Our work	$\tilde{\Omega}(n^{1-\frac{1}{k}})$	$k/2$	"rorrelation"

Optimal

Our results

Theorem.
Let k be any positive integer, $k \leq \frac{1}{3}\log n$. Then there is $f_k : \{-1,1\}^n \to \{0,1,*\}$ such that

$$Q_{\frac{1}{2}-\frac{1}{2^{k+4}}}(f_k) \leq \left\lceil \frac{k}{2} \right\rceil,$$

$$R_{\frac{1}{2^{k+1}}}(f_k) \geq \Omega\left(\frac{n^{1-\frac{1}{k}}}{(\log n)^{2-\frac{1}{k}}}\right).$$

Our results

Theorem.
Let k be any positive integer, $k \leq \frac{1}{3} \log n$. Then there is $f_k : \{-1,1\}^n \to \{0,1,*\}$ such that

$$Q_{\frac{1}{2}-\frac{1}{2^{k+4}}}(f_k) \leq \left\lceil \frac{k}{2} \right\rceil, \qquad Q_{1/3}(f_k) = O\left(k4^k\right),$$

$$R_{\frac{1}{2^{k+1}}}(f_k) \geq \Omega\left(\frac{n^{1-\frac{1}{k}}}{(\log n)^{2-\frac{1}{k}}}\right). \qquad R_{1/3}(f_k) = \Omega\left(\frac{n^{1-\frac{1}{k}}}{k(\log n)^{2-\frac{1}{k}}}\right).$$

Our results

Corollary 1.

For any $\epsilon > 0$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

$$Q_{1/3}(f) = O(1),$$

$$R_{1/3}(f) = \Omega(n^{1-\epsilon}) .$$

Our results

Corollary 1.

For any $\epsilon > 0$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

$$Q_{1/3}(f) = O(1),$$

$$R_{1/3}(f) = \Omega(n^{1-\epsilon}).$$

Take $k = 1 + \lceil 1/\epsilon \rceil$

Our results

Corollary 1.

For any $\epsilon > 0$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

$$Q_{1/3}(f) = O(1),$$
$$R_{1/3}(f) = \Omega(n^{1-\epsilon}).$$

Take $k = 1 + \lceil 1/\epsilon \rceil$

Corollary 2.

For any monotone $\alpha : \mathbb{N} \to \mathbb{N}$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

$$Q_{1/3}(f) \leq \alpha(n),$$
$$R_{1/3}(f) = n^{1-o(1)}.$$

Our results

Corollary 1.

For any $\epsilon > 0$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

$$Q_{1/3}(f) = O(1),$$
$$R_{1/3}(f) = \Omega(n^{1-\epsilon}).$$

Take $k = 1 + \lceil 1/\epsilon \rceil$

Corollary 2.

For any monotone $\alpha : \mathbb{N} \to \mathbb{N}$, there is $f : \{-1,1\}^n \to \{0,1,*\}$ with

$$Q_{1/3}(f) \leq \alpha(n),$$
$$R_{1/3}(f) = n^{1-o(1)}.$$

Take $k = k(n)$ an arbitrarily slow-growing function, e.g. $k = \log\log\log n$.

Our results: total functions

Reference	Randomized vs. Quantum
Grover 69, BBBV 97	$R(f) = \Omega(Q(f)^2)$
Beals et al. 01	$R(f) = O(Q(f)^6)$

Our results: total functions

Reference	Randomized vs. Quantum
Grover 69, BBBV 97	$R(f) = \Omega(Q(f)^2)$
Beals et al. 01	$R(f) = O(Q(f)^6)$
Aaronson et al. 16	$R(f) \geq \tilde{\Omega}(Q(f)^{2.5})$

Our results: total functions

Reference	Randomized vs. Quantum
Grover 69, BBBV 97	$R(f) = \Omega(Q(f)^2)$
Beals et al. 01	$R(f) = O(Q(f)^6)$
Aaronson et al. 16	$R(f) \geq \tilde{\Omega}(Q(f)^{2.5})$
Tal 19	$R(f) \geq Q(f)^{\frac{8}{3}-o(1)}$

Our results: total functions

Reference	Randomized vs. Quantum
Grover 69, BBBV 97	$R(f) = \Omega(Q(f)^2)$
Beals et al. 01	$R(f) = O(Q(f)^6)$
Aaronson et al. 16	$R(f) \geq \tilde{\Omega}(Q(f)^{2.5})$
Tal 19	$R(f) \geq Q(f)^{\frac{8}{3}-o(1)}$
Aaronson et al. 20	$R(f) = O(Q(f)^4)$

Our results: total functions

Reference	Randomized vs. Quantum
Grover 69, BBBV 97	$R(f) = \Omega(Q(f)^2)$
Beals et al. 01	$R(f) = O(Q(f)^6)$
Aaronson et al. 16	$R(f) \geq \tilde{\Omega}(Q(f)^{2.5})$
Tal 19	$R(f) \geq Q(f)^{\frac{8}{3}-o(1)}$
Aaronson et al. 20	$R(f) = O(Q(f)^4)$
Our work	$R(f) \geq Q(f)^{3-o(1)}$

Our results: total functions

Reference	Randomized vs. Quantum	
Grover 69, BBBV 97	$R(f) = \Omega(Q(f)^2)$	
Beals et al. 01	$R(f) = O(Q(f)^6)$	
Aaronson et al. 16	$R(f) \geq \tilde{\Omega}(Q(f)^{2.5})$	*“cheatsheet”*
Tal 19	$R(f) \geq Q(f)^{\frac{8}{3}-o(1)}$	*“cheatsheet”*
Aaronson et al. 20	$R(f) = O(Q(f)^4)$	
Our work	$R(f) \geq Q(f)^{3-o(1)}$	*“cheatsheet”*

Our results: communication

Partial functions $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1,*\}$,

Reference	Classical	Quantum
Buhrman et al. 98	$D(f) = \Omega(n)$	$O(\log n)$
Raz 99	$R(f) = \tilde{\Omega}(n^{1/4})$	$O(\log n)$
Klartag-Regev 10	$R(f) = \tilde{\Omega}(n^{1/3})$	$O(\log n)$
Aaronson-Ambainis 15	$R(f) = \tilde{\Omega}(n^{1/2})$	$O(\log n)$
Tal 19	$R(f) = \Omega(n^{2/3-\epsilon})$	$O(\log n)$
Our work	$R(f) = \Omega(n^{1-\epsilon})$	$O(\log n)$

Our results: communication

Partial functions $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1,*\}$,

Reference	Classical	Quantum
Buhrman et al. 98	$D(f) = \Omega(n)$	$O(\log n)$
Raz 99	$R(f) = \tilde{\Omega}(n^{1/4})$	$O(\log n)$
Klartag-Regev 10	$R(f) = \tilde{\Omega}(n^{1/3})$	$O(\log n)$
Aaronson-Ambainis 15	$R(f) = \tilde{\Omega}(n^{1/2})$	$O(\log n)$
Tal 19	$R(f) = \Omega(n^{2/3-\epsilon})$	$O(\log n)$
Our work	$R(f) = \Omega(n^{1-\epsilon})$	$O(\log n)$

near-optimal

Our results: communication

Partial functions $f: \{0,1\}^n \times \{0,1\}^n \to \{0,1,*\}$,

Reference	Classical	Quantum
Buhrman et al. 98	$D(f) = \Omega(n)$	$O(\log n)$
Raz 99	$R(f) = \tilde{\Omega}(n^{1/4})$	$O(\log n)$
Klartag-Regev 10	$R(f) = \tilde{\Omega}(n^{1/3})$	$O(\log n)$
Aaronson-Ambainis 15	$R(f) = \tilde{\Omega}(n^{1/2})$	$O(\log n)$
Tal 19	$R(f) = \Omega(n^{2/3-\epsilon})$	$O(\log n)$
Our work	$R(f) = \Omega(n^{1-\epsilon})$	$O(\log n)$

near-optimal

lifting from query model

Our results: communication

Total functions $f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,

Reference	Classical vs. Quantum
Buhrman et al. 98, Razborov 02	$R(f) \geq \Omega(Q(f)^2)$
Aaronson et al. 15	$R(f) \geq \tilde{\Omega}(Q(f)^{5/2})$
Tal 19	$R(f) \geq \Omega(Q(f)^{8/3-o(1)})$
Our work	$R(f) \geq \Omega(Q(f)^{3-o(1)})$

Our results: Fourier weight

Theorem

For any decision tree $g : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{g}(S)| \leq c^{\ell} \sqrt{\binom{d}{\ell} (1 + \log n)^{\ell - 1}} \,.$$

Our results: Fourier weight

Theorem

For any decision tree $g : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{g}(S)| \leq c^{\ell} \sqrt{\binom{d}{\ell}(1 + \log n)^{\ell-1}}.$$

- Essentially optimal

Our results: Fourier weight

Theorem

For any decision tree $g : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{g}(S)| \leq c^{\ell} \sqrt{\binom{d}{\ell} (1 + \log n)^{\ell - 1}} \,.$$

- Essentially optimal
- Settles conjecture by Tal (2019)

Our results: Fourier weight

Theorem

For any decision tree $g : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{g}(S)| \leq c^{\ell} \sqrt{\binom{d}{\ell}(1 + \log n)^{\ell-1}}.$$

- Essentially optimal
- Settles conjecture by Tal (2019)
- Previous bounds trivial already at $\ell \geq \sqrt{d}$

Independent work by Bansal & Sinha

Bansal–Sinha	**Our work**

Independent work by Bansal & Sinha

Bansal–Sinha

stochastic calculus

Our work

Fourier analysis

Independent work by Bansal & Sinha

Bansal–Sinha

stochastic calculus

- advanced machinery

Our work

Fourier analysis

- elementary

Independent work by Bansal & Sinha

Bansal–Sinha

stochastic calculus

- advanced machinery
- no Fourier weight bound

Our work

Fourier analysis

- elementary
- optimal Fourier weight of decision trees

Independent work by Bansal & Sinha

Bansal–Sinha

stochastic calculus

- advanced machinery
- no Fourier weight bound

explicit

Our work

Fourier analysis

- elementary
- optimal Fourier weight of decision trees

existential

The problem: rorrelation

Rorrelation

$U \in \mathbb{R}^{n \times n}$, orthogonal matrix

$x_1, x_2, \ldots, x_k \in \{-1, 1\}^n$

Rorrelation

$U \in \mathbb{R}^{n \times n}$, orthogonal matrix

$x_1, x_2, \ldots, x_k \in \{-1,1\}^n$

$$\phi_{n,k,U}(x_1, x_2, \ldots, x_k) = \frac{1}{n}\mathbf{1}^T D_{x_1} U D_{x_2} U \cdots U D_{x_k} \mathbf{1}$$

Rorrelation

$U \in \mathbb{R}^{n\times n}$, orthogonal matrix

$x_1, x_2, \ldots, x_k \in \{-1,1\}^n$

$$\phi_{n,k,U}(x_1, x_2, \ldots, x_k) = \frac{1}{n}\mathbf{1}^T D_{x_1} U D_{x_2} U \cdots U D_{x_k} \mathbf{1}$$

$$f_{n,k,U}(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \phi_{n,k,U} > 2^{-k}, \\ 0 & |\phi_{n,k,U}| \leq 2^{-k-1}, \\ * & \text{otherwise}. \end{cases}$$

Rorrelation: quantum algorithms

$$\phi_{n,k,U}(x_1, x_2, \ldots, x_k) = \frac{1}{n}\mathbf{1}^T D_{x_1} U D_{x_2} U \cdots U D_{x_k} \mathbf{1}$$

$$f_{n,k,U}(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \phi_{n,k,U} > 2^{-k}, \\ 0 & |\phi_{n,k,U}| \leq 2^{-k-1}, \\ * & \text{otherwise}. \end{cases}$$

Rorrelation: quantum algorithms

$$\phi_{n,k,U}(x_1, x_2, \ldots, x_k) = \frac{1}{n}\mathbf{1}^T D_{x_1} U D_{x_2} U \cdots U D_{x_k} \mathbf{1}$$

$$f_{n,k,U}(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \phi_{n,k,U} > 2^{-k}, \\ 0 & |\phi_{n,k,U}| \le 2^{-k-1}, \\ * & \text{otherwise}\,. \end{cases}$$

Theorem (Aaronson-Ambainis, Tal).
There is a quantum algorithm using $\lceil k/2 \rceil$ queries that accepts x with probability

$$\frac{\phi_{n,k,U}(x) + 1}{2}\,.$$

Rorrelation: classical lower bound

—the "indistinguishability" argument

Rorrelation: classical lower bound

—the "indistinguishability" argument

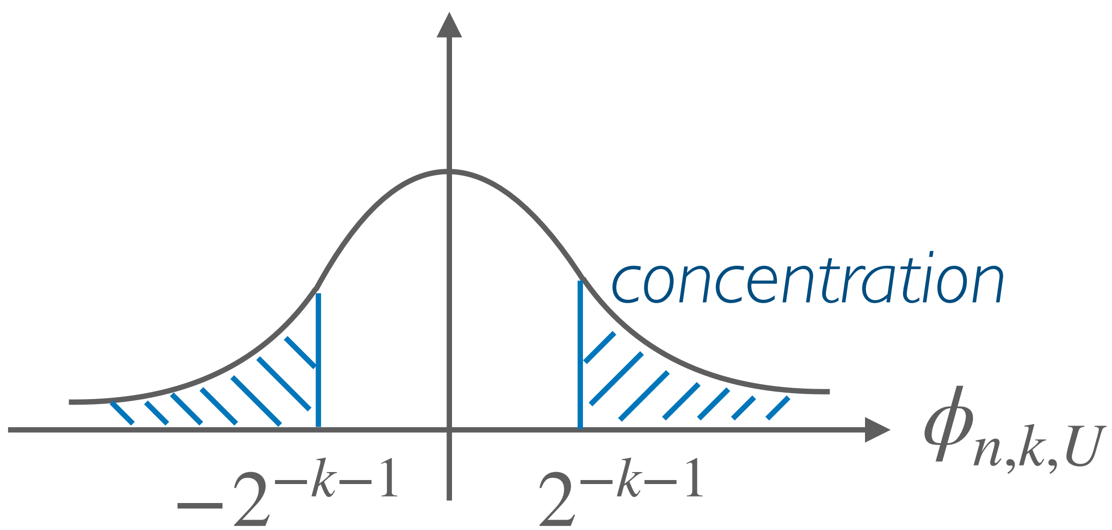
$\mathscr{U}_{n,k}$ = uniform distribution

$\mathscr{D}_{n,k,U}$ = correlated distribution

Rorrelation: classical lower bound
—the "indistinguishability" argument

$\mathscr{U}_{n,k}$ = uniform distribution

$\mathscr{D}_{n,k,U}$ = correlated distribution

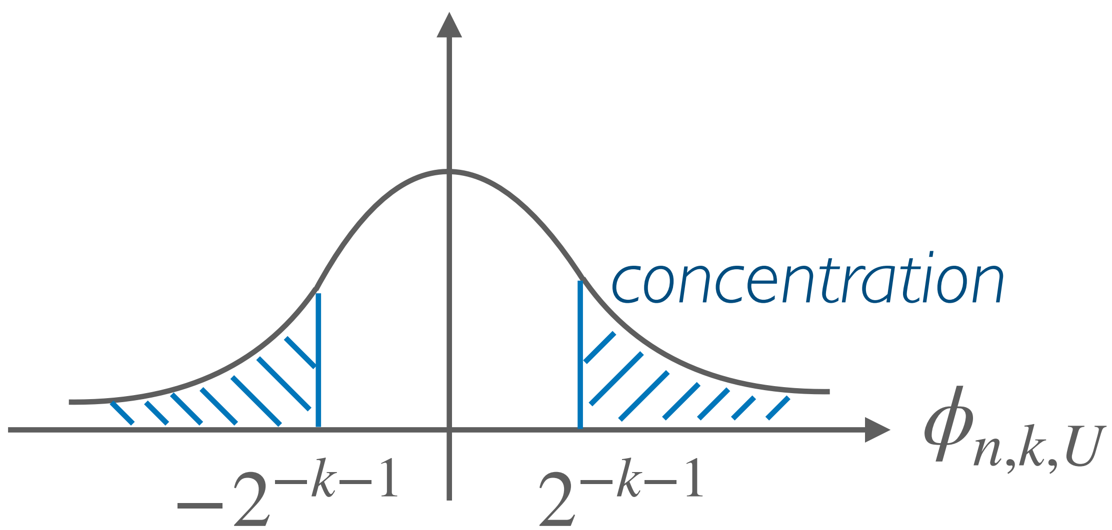


$$\mathbf{P}_{\mathscr{U}_{n,k}}[\phi > 2^{-k-1}] < 2^{-k-1}$$

Rorrelation: classical lower bound

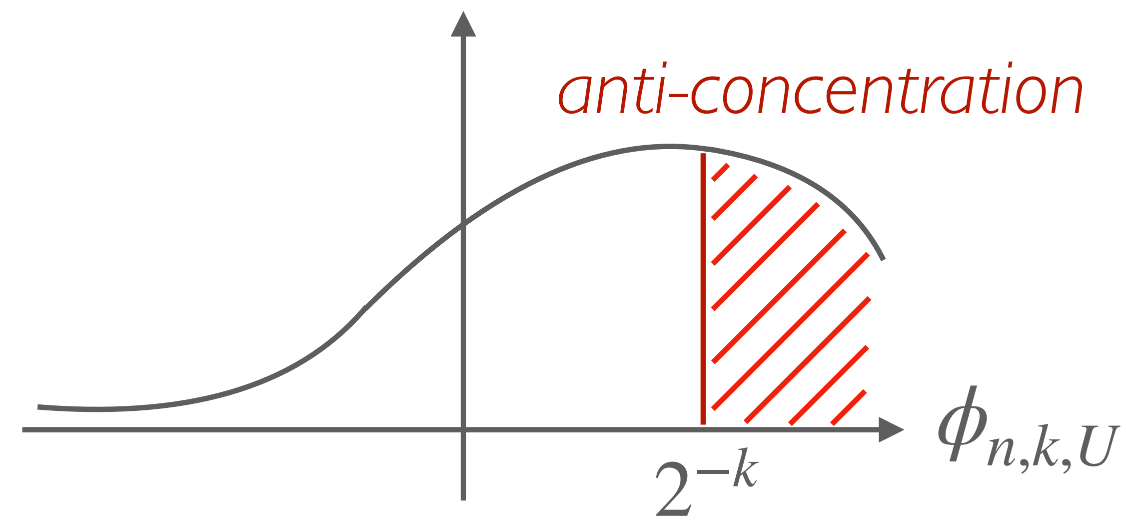
—the "indistinguishability" argument

$\mathscr{U}_{n,k}$ = uniform distribution



$$\mathbf{P}_{\mathscr{U}_{n,k}}[\phi > 2^{-k-1}] < 2^{-k-1}$$

$\mathscr{D}_{n,k,U}$ = correlated distribution

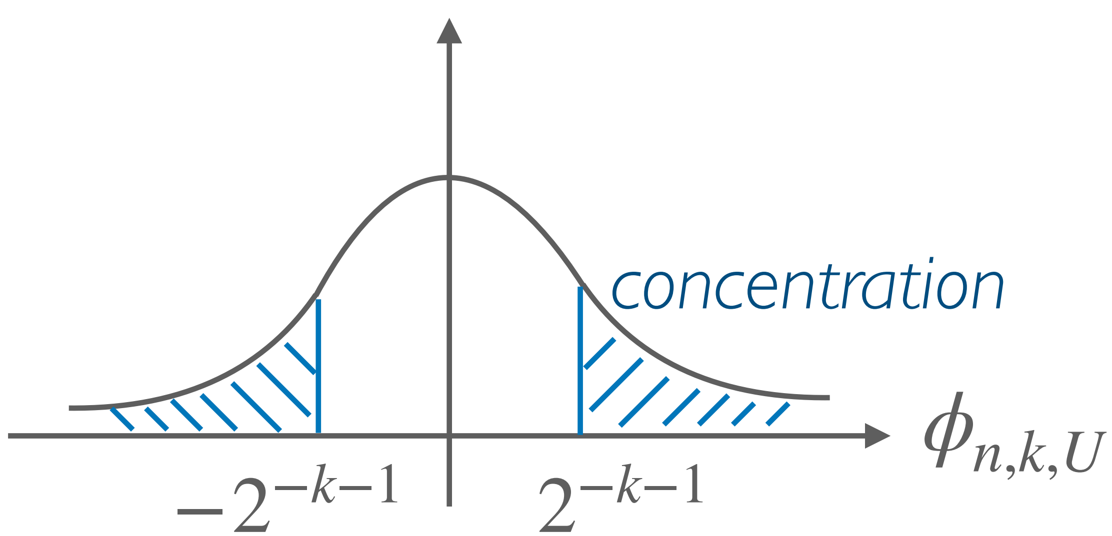


$$\mathbf{P}_{\mathscr{D}_{n,k,U}}[\phi \geq 2^{-k}] \geq 2^{-k}$$

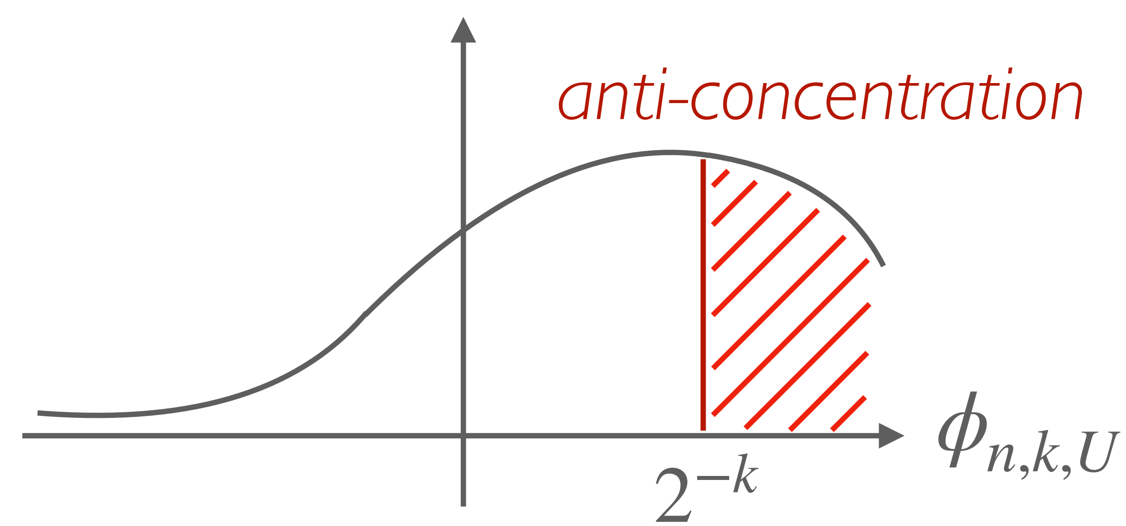
Rorrelation: classical lower bound

—the "indistinguishability" argument

$\mathscr{U}_{n,k}$ = uniform distribution



$\mathscr{D}_{n,k,U}$ = correlated distribution



Thus, for any randomized query algorithm g of error ϵ,

$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) \ - \ \mathbf{E}_{\mathscr{U}_{n,k}} g(x) \ \geq 2^{-k-1} - 2\epsilon \, .$$

Rorrelation: classical lower bound
—the "indistinguishability" argument

$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x)$$

Rorrelation: classical lower bound

—the "indistinguishability" argument

$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x)$$

$$\leq \left| \mathbf{E}_{\mathscr{D}_{n,k,U}} \sum_S \hat{g}(S)\chi_S - \mathbf{E}_{\mathscr{U}_{n,k}} \sum_S \hat{g}(S)\chi_S \right|$$

Rorrelation: classical lower bound
—the "indistinguishability" argument

$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x)$$

$$\leq \left| \mathbf{E}_{\mathscr{D}_{n,k,U}} \sum_{S} \hat{g}(S)\chi_S - \mathbf{E}_{\mathscr{U}_{n,k}} \sum_{S} \hat{g}(S)\chi_S \right|$$

$$= \left| \sum_{S \neq \varnothing} \hat{g}(S) \mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S \right|$$

Rorrelation: classical lower bound

—the "indistinguishability" argument

$$
\begin{aligned}
&\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x) \\
&\qquad \le \left| \mathbf{E}_{\mathscr{D}_{n,k,U}} \sum_{S} \hat{g}(S)\chi_S - \mathbf{E}_{\mathscr{U}_{n,k}} \sum_{S} \hat{g}(S)\chi_S \right| \\
&\qquad = \left| \sum_{S \neq \varnothing} \hat{g}(S) \mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S \right| \\
&\qquad \le \sum_{\ell=1}^{n} \sum_{|S|=\ell} |\hat{g}(S)| \cdot \max_{T=\ell} |\mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S|
\end{aligned}
$$

Rorrelation: classical lower bound

—the "indistinguishability" argument

$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x)$$

$$\leq \left| \mathbf{E}_{\mathscr{D}_{n,k,U}} \sum_{S} \hat{g}(S)\chi_S - \mathbf{E}_{\mathscr{U}_{n,k}} \sum_{S} \hat{g}(S)\chi_S \right|$$

$$= \left| \sum_{S \neq \varnothing} \hat{g}(S)\mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S \right|$$

$$\leq \sum_{\cancel{\ell=1}}^{n} \sum_{|S|=\ell} |\hat{g}(S)| \cdot \max_{T=\ell} |\mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S|$$

$\ell = k$

[Tal]

$$\leq O\left(\frac{\ell \log n}{n}\right)^{\frac{\ell}{2}\frac{k-1}{k}}$$

Rorrelation: classical lower bound

—the “indistinguishability” argument

$$
\begin{aligned}
&\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x) \\
&\qquad \le \left| \mathbf{E}_{\mathscr{D}_{n,k,U}} \sum_S \hat{g}(S)\chi_S - \mathbf{E}_{\mathscr{U}_{n,k}} \sum_S \hat{g}(S)\chi_S \right| \\
&\qquad = \left| \sum_{S \neq \varnothing} \hat{g}(S) \mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S \right| \\
&\qquad \le \sum_{\cancel{\ell=1}}^{n} \sum_{|S|=\ell} |\hat{g}(S)| \cdot \max_{T=\ell} |\mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S|
\end{aligned}
$$

$\ell = k$

[Tal]

$$\le O\left(\frac{\ell \log n}{n}\right)^{\frac{\ell}{2}\frac{k-1}{k}}$$

We prove: $\le c^{\ell} \sqrt{\binom{d}{\ell} (\ln en)^{\ell-1}}$

Rorrelation: classical lower bound

—the "indistinguishability" argument

$$\mathbf{E}_{\mathscr{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathscr{U}_{n,k}} g(x)$$

$$\leq \left| \mathbf{E}_{\mathscr{D}_{n,k,U}} \sum_S \hat{g}(S)\chi_S - \mathbf{E}_{\mathscr{U}_{n,k}} \sum_S \hat{g}(S)\chi_S \right|$$

$$= \left| \sum_{S \neq \varnothing} \hat{g}(S) \mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S \right|$$

$$\leq \sum_{\cancel{\ell=1}}^{n} \sum_{|S|=\ell} |\hat{g}(S)| \cdot \max_{T=\ell} |\mathbf{E}_{\mathscr{D}_{n,k,U}} \chi_S|$$

$\ell = k$

[Tal]

$$\leq O\left(\frac{\ell \log n}{n}\right)^{\frac{\ell}{2}\frac{k-1}{k}}$$

We prove: $\leq c^{\ell} \sqrt{\binom{d}{\ell}} \, (\ln en)^{\ell-1}$

Therefore,

$$R_{2-o(k)}(f_k) = \tilde{\Omega}(n^{1-\frac{1}{k}}) . \quad \blacksquare$$

Fourier weight of decision trees

Fourier weight of decision trees

Main Theorem.

For any decision tree $T : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{T}(S)| \leq c^{\ell} \sqrt{\binom{d}{\ell} (1 + \log n)^{\ell-1}} .$$

Fourier weight of decision trees

Main Theorem.

For any decision tree $T : \{-1,1\}^n \to \{0,1\}$ of depth d,

$$\||L_\ell T\|| \qquad \sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{T}(S)| \le c^\ell \sqrt{\binom{d}{\ell}(1+\log n)^{\ell-1}}\,.$$

Fourier weight of decision trees

Main Theorem.

Fix any decision tree $T : \{-1,1\}^n \to \{-1,0,1\}$ of depth d, and $\mathbf{P}[T(x) \neq 0] = p$. Then

$$\|\| L_\ell T \|\| \leq c^\ell \sqrt{\binom{d}{\ell}} \Lambda_{n^2,\ell}(p),$$

Fourier weight of decision trees

Main Theorem.

Fix any decision tree $T : \{-1,1\}^n \to \{-1,0,1\}$ of depth d, and $\mathrm{dns}(T) = p$. Then

$$|||L_\ell T||| \leq c^\ell \sqrt{\binom{d}{\ell}} \Lambda_{n^2,\ell}(p),$$

$$\Lambda_{m,\ell} = \begin{cases} 0, & \textit{if } p = 0, \\ p\sqrt{\left(\frac{1}{\ell}\ln\frac{e^\ell m^{\ell-1}}{p}\right)^\ell}, & \textit{if } 0 < p \leq 1/m, \\ p\sqrt{\left(\ln\frac{e}{p}\right)(\ln em)^{\ell-1}}, & \textit{if } 1/m \leq p \leq 1. \end{cases}$$

Fourier weight of decision trees

Main Theorem.
Fix any decision tree $T : \{-1,1\}^n \to \{-1,0,1\}$ of depth d, and $\mathrm{dns}(T) = p$. Then

$$|||L_\ell T||| \le c^\ell \sqrt{\binom{d}{\ell}} \Lambda_{n^2,\ell}(p),$$

$$\Lambda_{m,\ell} = \begin{cases} 0, & \textit{if } p = 0, \\ p\sqrt{\left(\frac{1}{\ell}\ln\frac{e^\ell m^{\ell-1}}{p}\right)^\ell}, & \textit{if } 0 < p \le 1/m, \\ p\sqrt{\left(\ln\frac{e}{p}\right)(\ln em)^{\ell-1}}, & \textit{if } 1/m \le p \le 1. \end{cases}$$

increasing, concave

Fourier weight of decision trees

Main Theorem.

Fix any decision tree $T : \{-1,1\}^n \to \{-1,0,1\}$ of depth d, and $\mathrm{dns}(T) = p$. Then

$$\| L_\ell T \| \leq c^\ell \sqrt{\binom{d}{\ell} \Lambda_{n^2,\ell}(p)}, \quad \leq \sqrt{(\ln(en^2))^{\ell-1}}$$

$$\Lambda_{m,\ell} = \begin{cases} 0, & \textit{if } p = 0, \\ p\sqrt{\left(\frac{1}{\ell}\ln\frac{e^\ell m^{\ell-1}}{p}\right)^\ell}, & \textit{if } 0 < p \leq 1/m, \\ p\sqrt{\left(\ln\frac{e}{p}\right)(\ln em)^{\ell-1}}, & \textit{if } 1/m \leq p \leq 1. \end{cases}$$

increasing, concave

Known results

Nonadaptive case:

$$|||L_\ell T||| \le \sqrt{\binom{d}{\ell} \sum_{S:|S|=\ell} \hat{T}(S)^2} \le \sqrt{\binom{d}{\ell}}\,.$$

Known results

Nonadaptive case:

$$|||L_\ell T||| \leq \sqrt{\binom{d}{\ell}\sum_{S:|S|=\ell}\hat{T}(S)^2} \leq \sqrt{\binom{d}{\ell}}\,.$$

Cauchy–Schwarz

Known results

Nonadaptive case:

$$|||\, L_\ell T \,||| \le \sqrt{\binom{d}{\ell} \sum_{S:|S|=\ell} \hat{T}(S)^2} \le \sqrt{\binom{d}{\ell}}\,.$$

Cauchy–Schwarz

Parseval's identity

Known results

Adaptive, $\ell = 1$:

Known results

Adaptive, $\ell = 1$:

By negating variables, we can assume $\hat{T}(i) \geq 0$. Then

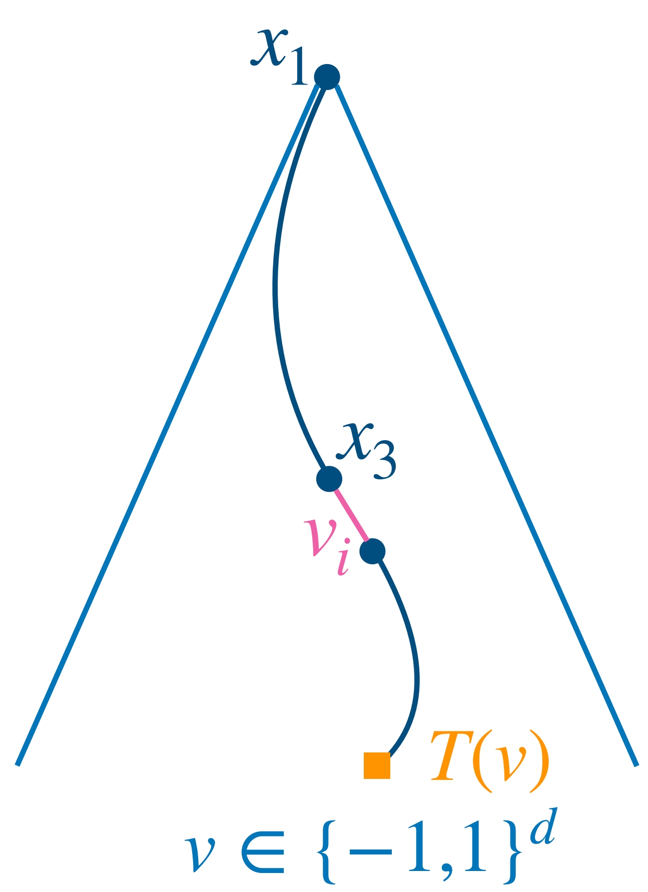
$$|||\, L_1\, T \,||| = \sum_{i=1}^{n} \hat{T}(i) = \mathbf{E}_{v \in \{-1,1\}^d} \left[T(v) \sum_{i=1}^{d} v_i \right]$$

Known results

Adaptive, $\ell = 1$:

By negating variables, we can assume $\hat{T}(i) \geq 0$. Then

$$|||L_1 T||| = \sum_{i=1}^{n} \hat{T}(i) = \mathbf{E}_{v \in \{-1,1\}^d} \left[T(v) \sum_{i=1}^{d} v_i \right]$$

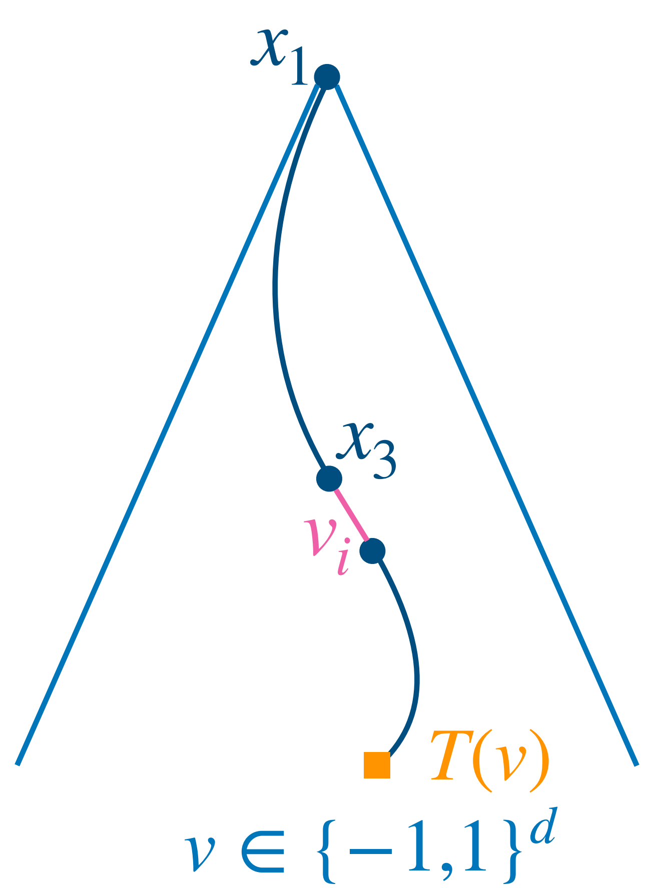


Known results

Adaptive, $\ell = 1$:

By negating variables, we can assume $\hat{T}(i) \geq 0$. Then

$$|||L_1 T||| = \sum_{i=1}^{n} \hat{T}(i) = \mathbf{E}_{v \in \{-1,1\}^d} \left[T(v) \sum_{i=1}^{d} v_i \right]$$



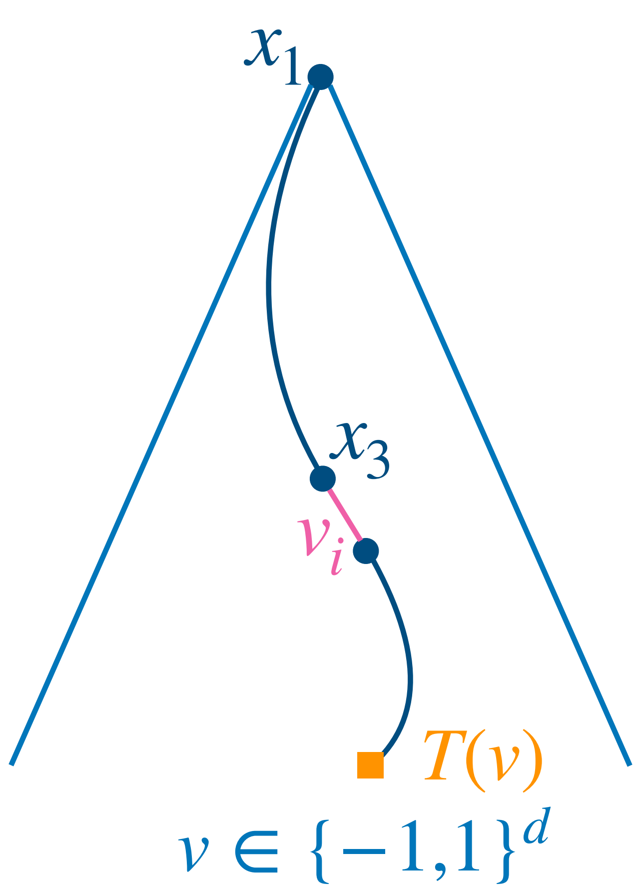
Maximized when paths with larger sums have label 1

Known results

Adaptive, $\ell = 1$:

By negating variables, we can assume $\hat{T}(i) \geq 0$. Then

$$|||L_1 T||| = \sum_{i=1}^{n} \hat{T}(i) = \mathbf{E}_{v \in \{-1,1\}^d} \left[T(v) \sum_{i=1}^{d} v_i \right]$$

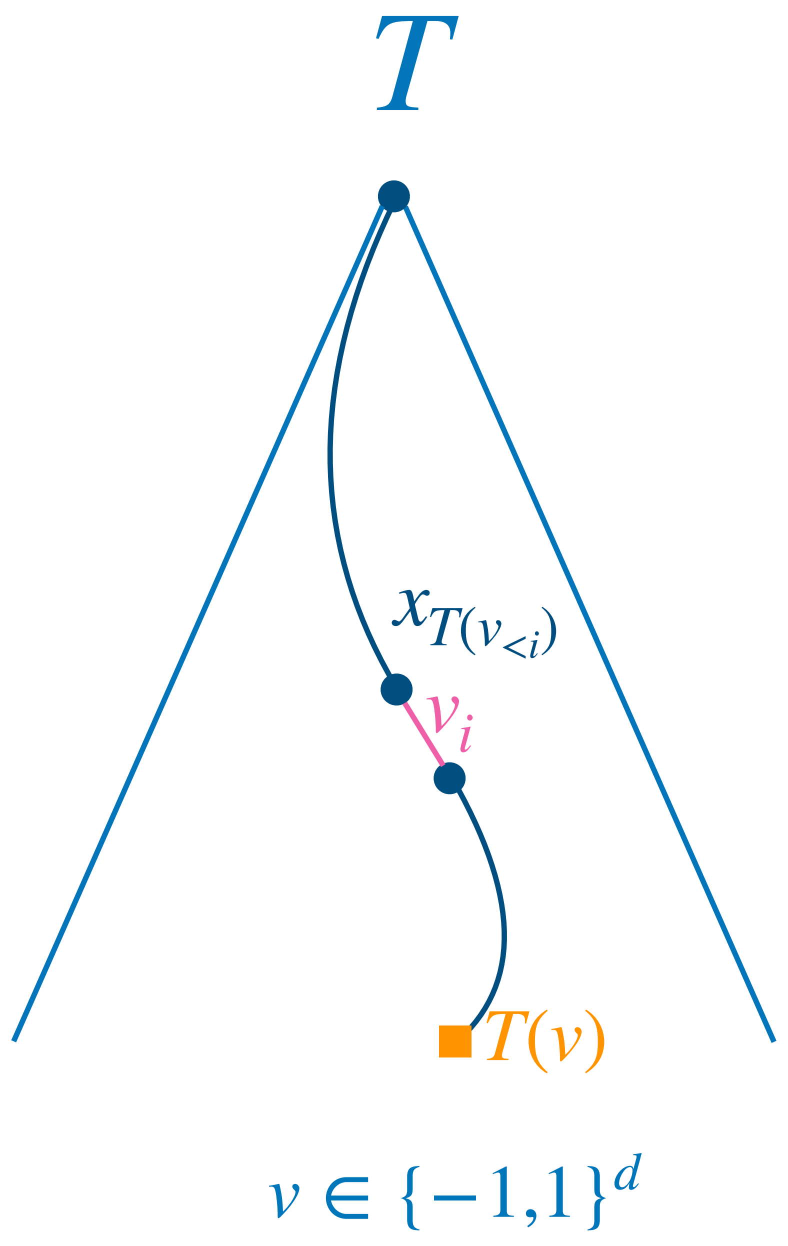


Theorem (O'Donnel-Servedio 07, Tal 19)

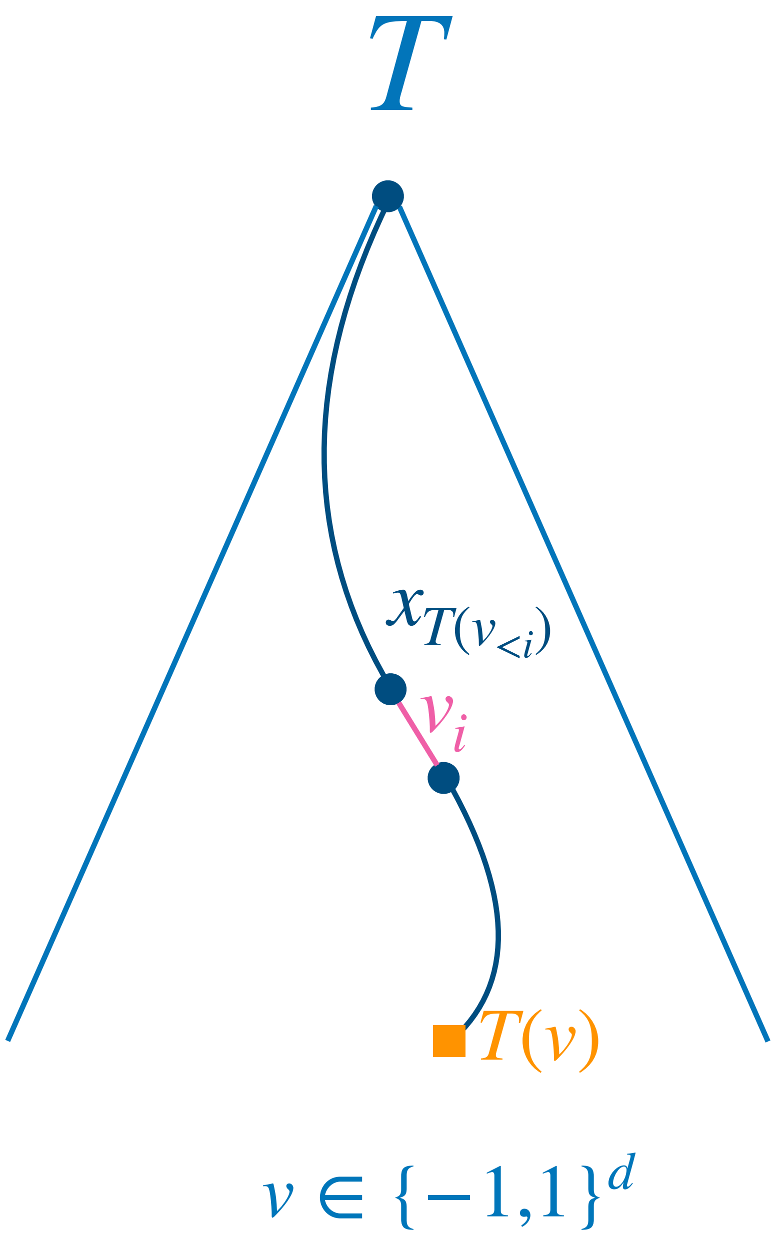
$$|||L_1 T||| \leq O\left(p\sqrt{d \ln \frac{e}{p}}\right).$$

Our approach

Our approach

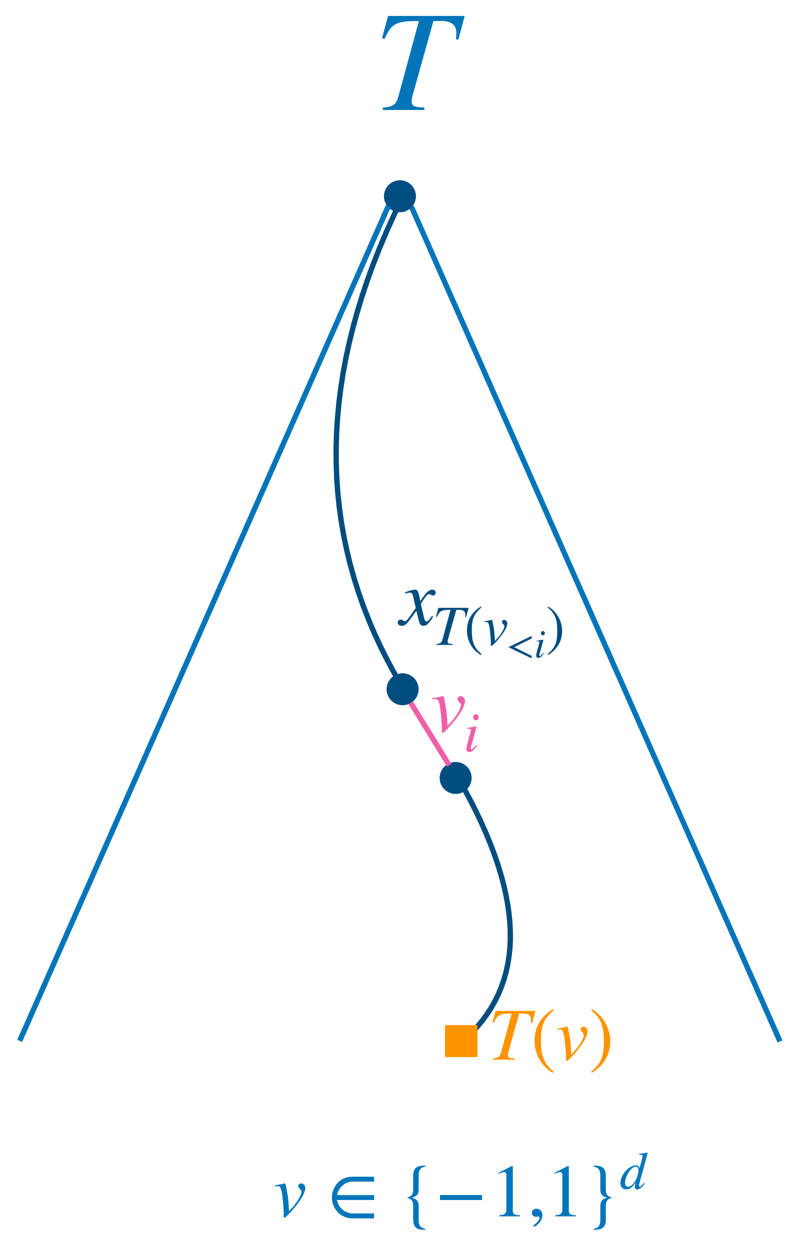


Our approach



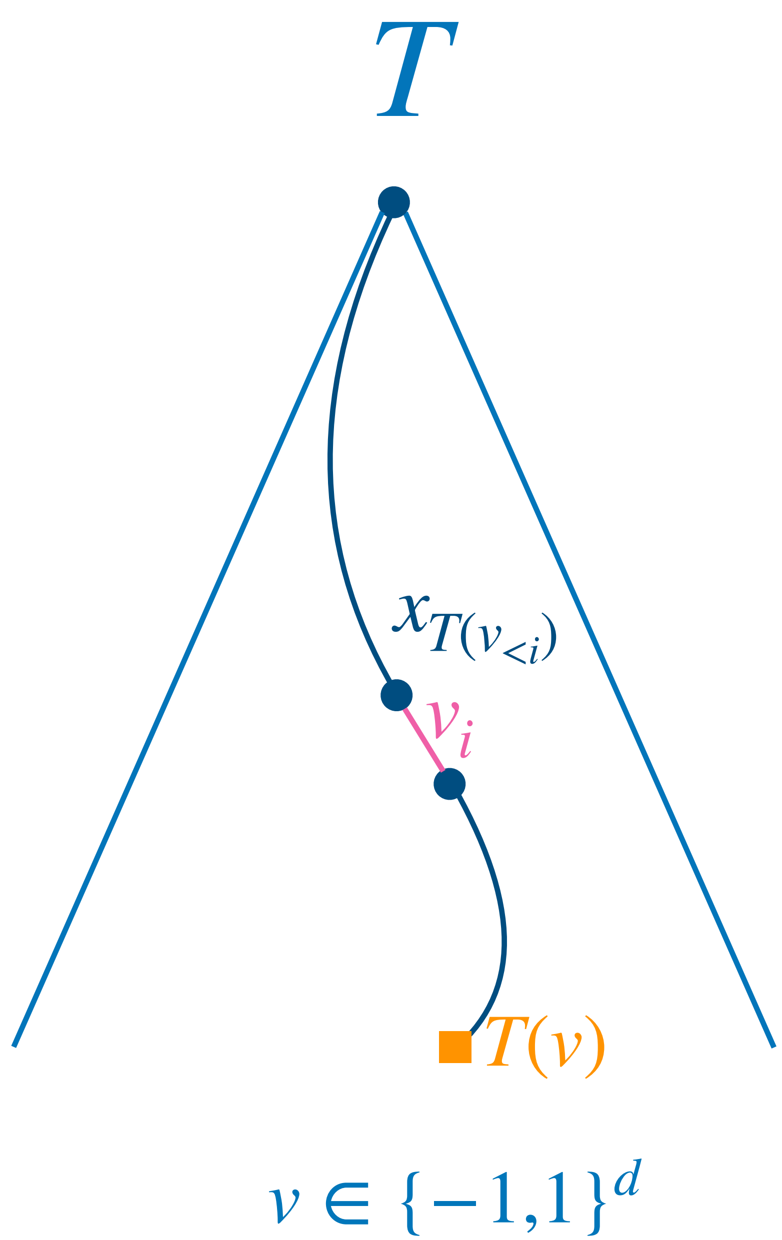
$$T(v) \prod_{i=1}^{d} \frac{1 + v_i x_{T(v_{<i})}}{2}$$

Our approach



$$T = \sum_{v \in \{-1,1\}^d} T(v) \prod_{i=1}^{d} \frac{1 + v_i x_{T(v_{<i})}}{2}$$

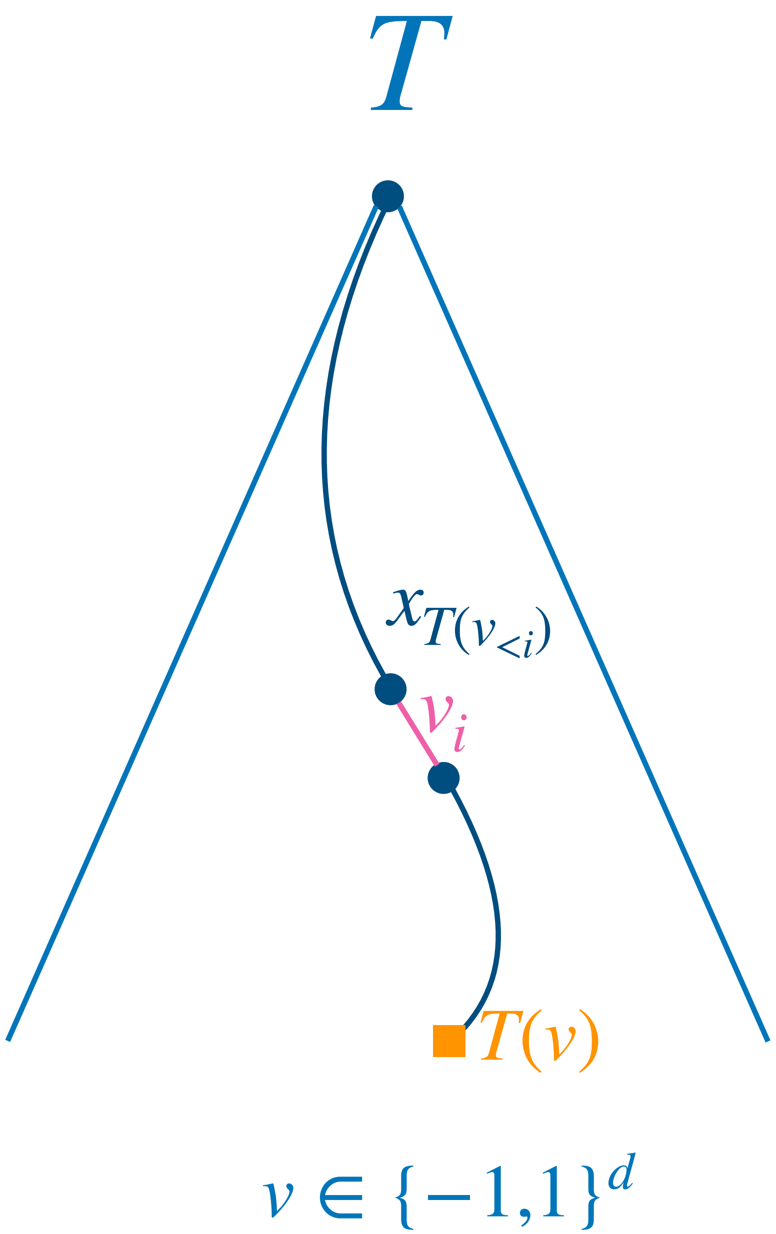
Our approach



$$T = \sum_{v \in \{-1,1\}^d} T(v) \prod_{i=1}^{d} \frac{1 + v_i x_{T(v_{<i})}}{2}$$

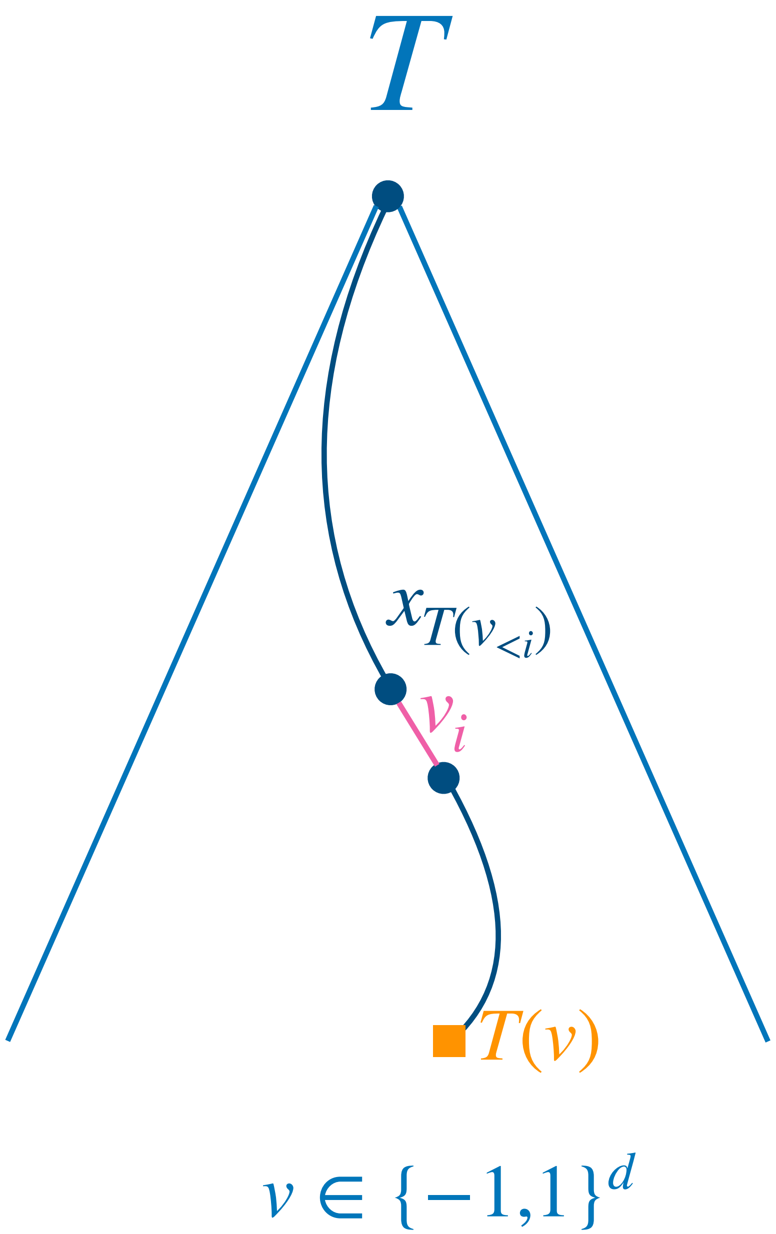
$$= \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \sum_{S \subseteq \{1,\ldots,d\}} \prod_{i \in S} v_i x_{T(v_{<i})}$$

Our approach



$$
\begin{aligned}
T &= \sum_{v\in\{-1,1\}^d} T(v) \prod_{i=1}^{d} \frac{1 + v_i x_{T(v_{<i})}}{2} \\
&= \sum_{v\in\{-1,1\}^d} T(v) 2^{-d} \sum_{S\subseteq\{1,\ldots,d\}} \prod_{i\in S} v_i x_{T(v_{<i})} \\
&= \sum_{S\subseteq\{1,\ldots,d\}} \sum_{v\in\{-1,1\}^d} T(v) 2^{-d} \prod_{i\in S} v_i x_{T(v_{<i})} .
\end{aligned}
$$

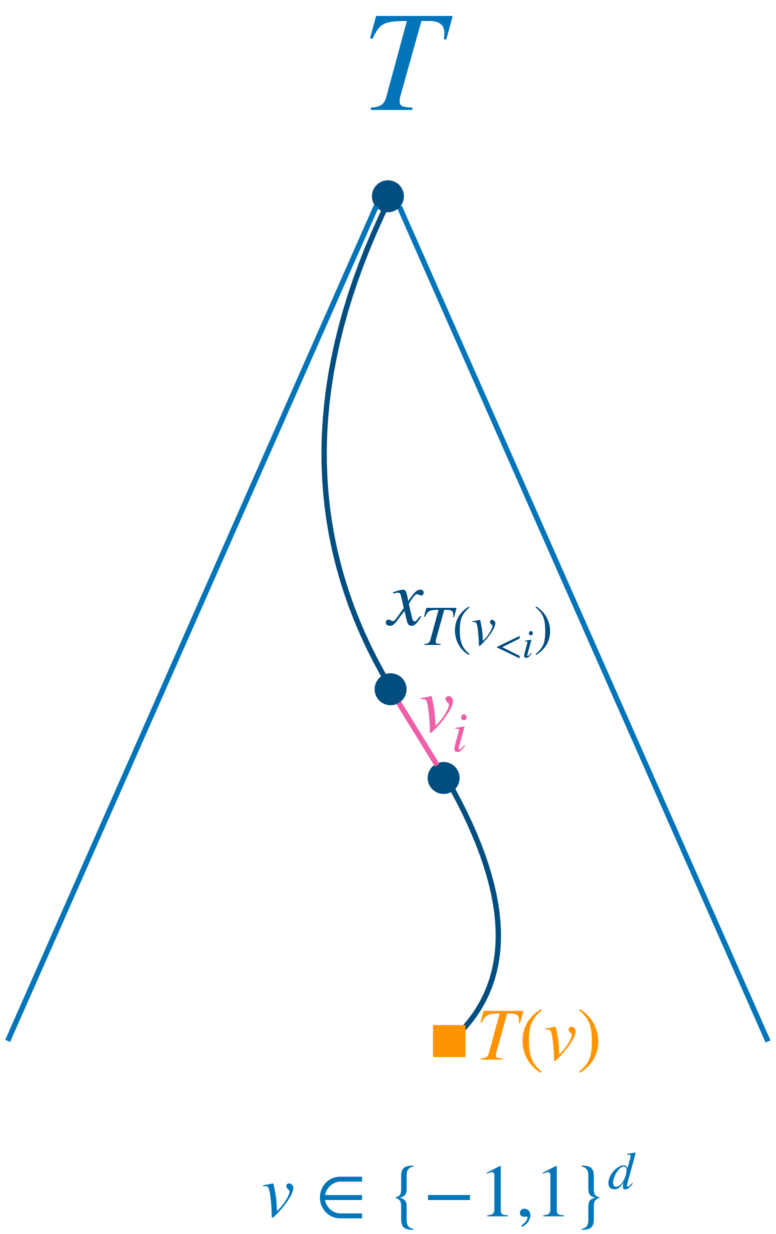
Our approach



$$T = \sum_{v\in\{-1,1\}^d} T(v)\prod_{i=1}^{d}\frac{1+v_i x_{T(v_{<i})}}{2}$$
$$= \sum_{v\in\{-1,1\}^d} T(v)2^{-d}\sum_{S\subseteq\{1,\ldots,d\}}\prod_{i\in S} v_i x_{T(v_{<i})}$$
$$= \sum_{S\subseteq\{1,\ldots,d\}}\sum_{v\in\{-1,1\}^d} T(v)2^{-d}\prod_{i\in S} v_i x_{T(v_{<i})}\,.$$

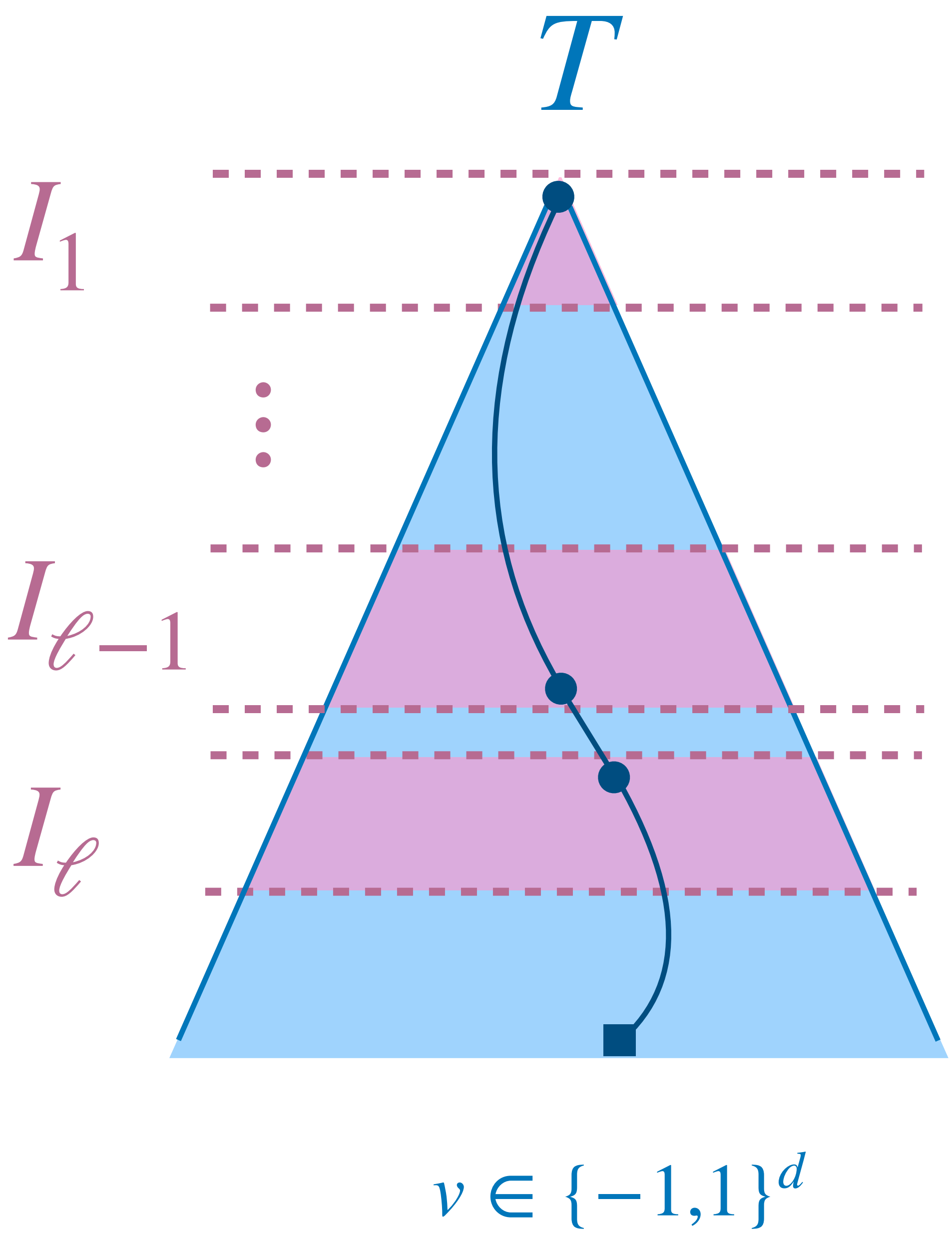
$$L_\ell T = \sum_{S\in\mathscr{P}_{d,\ell}}\sum_{v\in\{-1,1\}^d} T(v)2^{-d}\prod_{i\in S} v_i x_{T(v_{<i})}\,.$$

Our approach



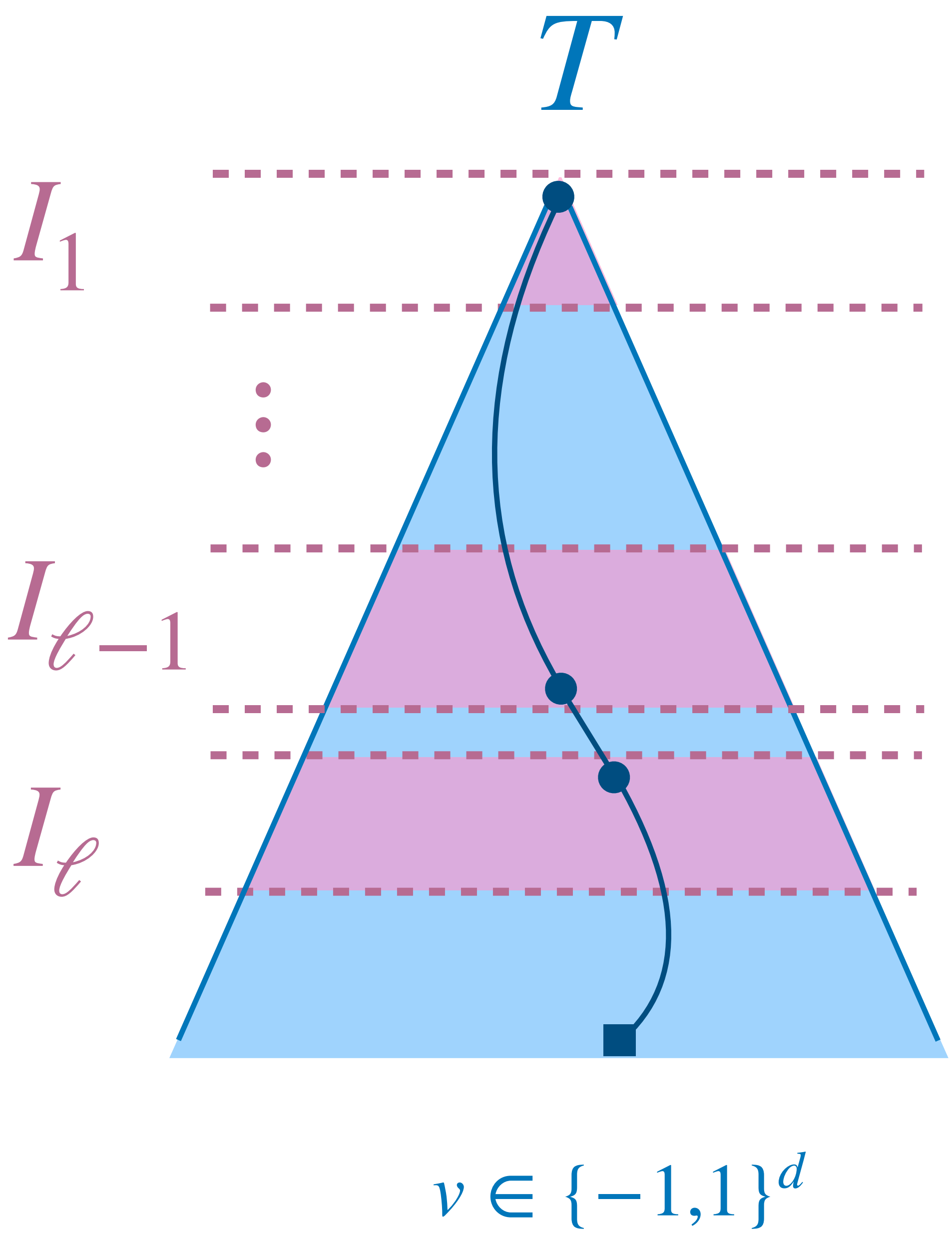
$$L_\ell T = \sum_{S \in \mathscr{P}_{d,\ell}} \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} \,.$$

Our approach



$$L_\ell T = \sum_{S \in \mathscr{P}_{d,\ell}} \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} .$$

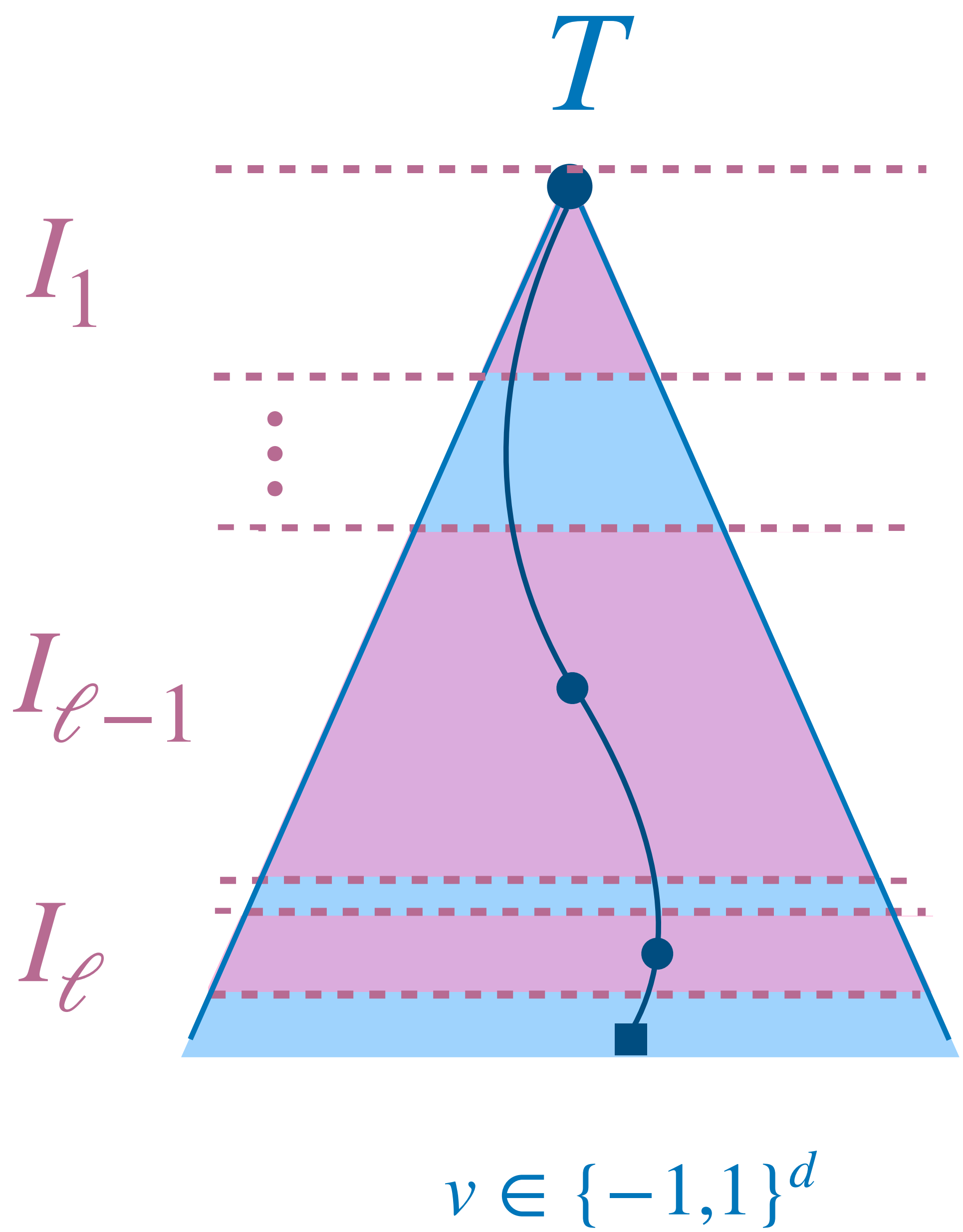
Our approach



$$L_\ell T = \sum_{S \in \mathscr{P}_{d,\ell}} \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} \,.$$

$$T|_{I_1 * I_2 * \dots * I_\ell} = \sum_{\substack{S \subseteq \{1,\dots,d\}: \\ |S \cap I_i| = 1}} \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} \,.$$

Our approach

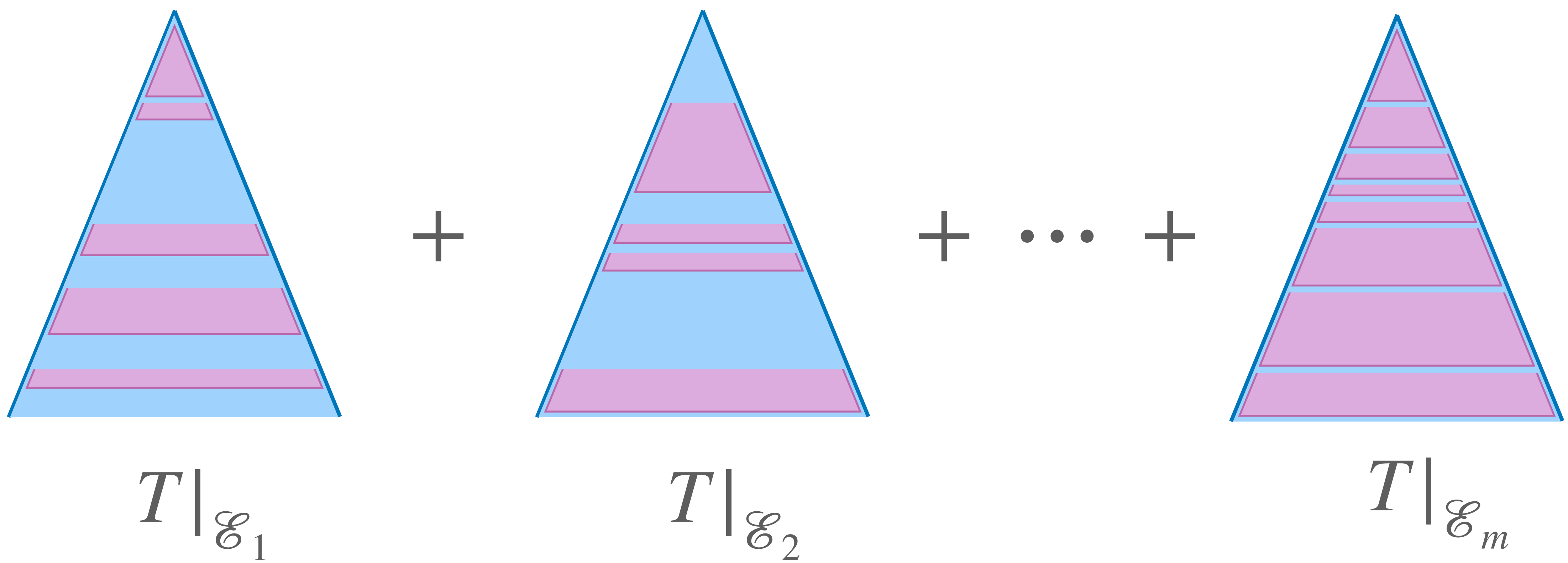


$$L_\ell T = \sum_{S \in \mathscr{P}_{d,\ell}} \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} \,.$$

$$T|_{I_1 * I_2 * \ldots * I_\ell} = \sum_{\substack{S \subseteq \{1,\ldots,d\}: \\ |S \cap I_i| = 1}} \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} \,.$$

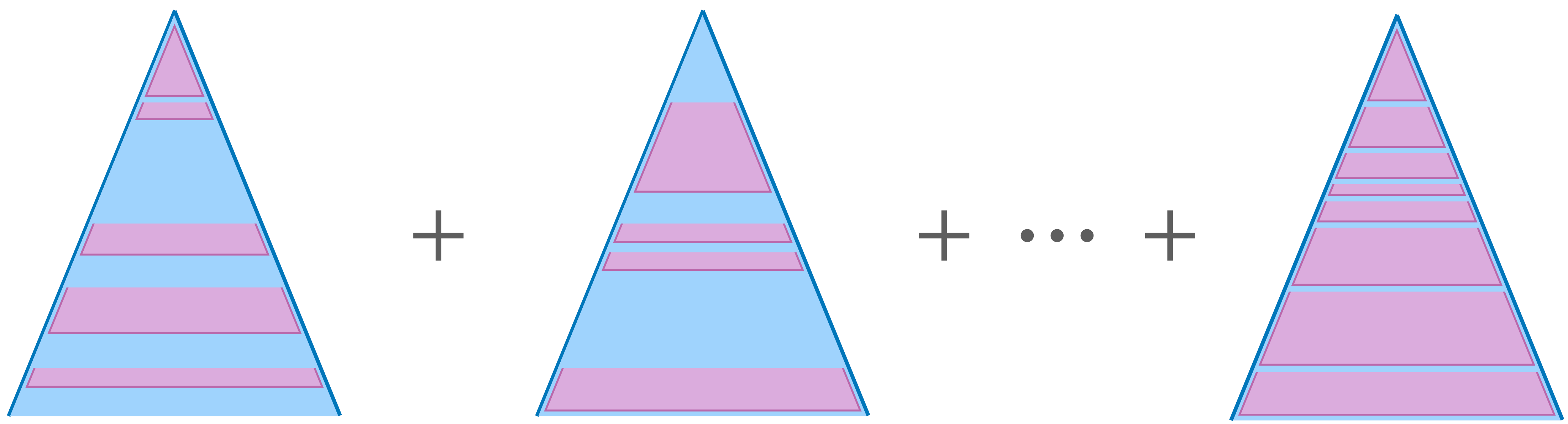
Our key idea

$L_\ell T =$



Our key idea

$L_\ell T =$



$T|_{\mathscr{E}_1} \qquad + \qquad T|_{\mathscr{E}_2} \qquad + \cdots + \qquad T|_{\mathscr{E}_m}$

$\Vert L_\ell T \Vert \leq \sum \Vert T|_{\mathscr{E}_i} \Vert$. (Triangle-inequality)

Our proof

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert L_\ell T \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert \leq \sum_i \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert T|_{\mathscr{E}_i} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert .$$

Our proof

$$\|\|\| L_\ell T \|\|\| \le \sum_i \|\|\| T|_{\mathscr{E}_i} \|\|\| \,.$$

Theorem 1.

For some absolute constant c, and any elementary family $\mathscr{E} = I_1 * I_2 * \cdots * I_\ell$,

$$\|\|\| T|_{\mathscr{E}} \|\|\| \le c^\ell \sqrt{|\mathscr{E}|}\, \Lambda_{n^2,\ell}(\mathrm{dns}(T)) \,.$$

Our proof

$$|||L_\ell T||| \le \sum_i |||T|_{\mathscr{E}_i}|||.$$

Theorem 1.

For some absolute constant c, and any elementary family $\mathscr{E} = I_1 * I_2 * \cdots * I_\ell$,

$$|||T|_{\mathscr{E}}||| \le c^\ell \sqrt{|\mathscr{E}|}\,\Lambda_{n^2,\ell}(\mathrm{dns}(T)).$$

Theorem 2.

$\mathscr{P}_{d,\ell}$ can be partitioned into elementary families $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_m$ s.t. for some const C,

$$\sum_{i=1}^{m} \sqrt{|\mathscr{E}_i|} \le C^\ell \sqrt{\binom{d}{\ell}}.$$

Our proof

$$|||L_\ell T||| \le \sum_i |||T|_{\mathscr{E}_i}||| \,.$$

Theorem 1.

For some absolute constant c, and any elementary family $\mathscr{E} = I_1 * I_2 * \cdots * I_\ell$,

$$|||T|_{\mathscr{E}}||| \le c^\ell \sqrt{|\mathscr{E}|}\,\Lambda_{n^2,\ell}(\mathrm{dns}(T))\,.$$

Theorem 2.

$\mathscr{P}_{d,\ell}$ can be partitioned into elementary families $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_m$ s.t. for some const C,

$$\sum_{i=1}^m \sqrt{|\mathscr{E}_i|} \le C^\ell \sqrt{\binom{d}{\ell}}\,.$$

$$|||L_\ell T||| \le (cC)^\ell \sqrt{\binom{d}{\ell}} \times \Lambda_{n^2,\ell}(\mathrm{dns}(T))$$

$\blacksquare$

Our proof

$$\|\!| L_\ell T |\!\|\ \le \sum_i \|\!| T|_{\mathscr{E}_i} |\!\|\,.$$

Theorem 1.

For some absolute constant c, and any elementary family $\mathscr{E} = I_1 * I_2 * \cdots * I_\ell$,

$$\|\!| T|_{\mathscr{E}} |\!\| \le c^\ell \sqrt{|\mathscr{E}|}\,\Lambda_{n^2,\ell}(\mathrm{dns}(T))\,.$$

Theorem 2. ⚠ **not exactly**

$\mathscr{P}_{d,\ell}$ can be partitioned into elementary families $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_m$ s.t. for some const C,

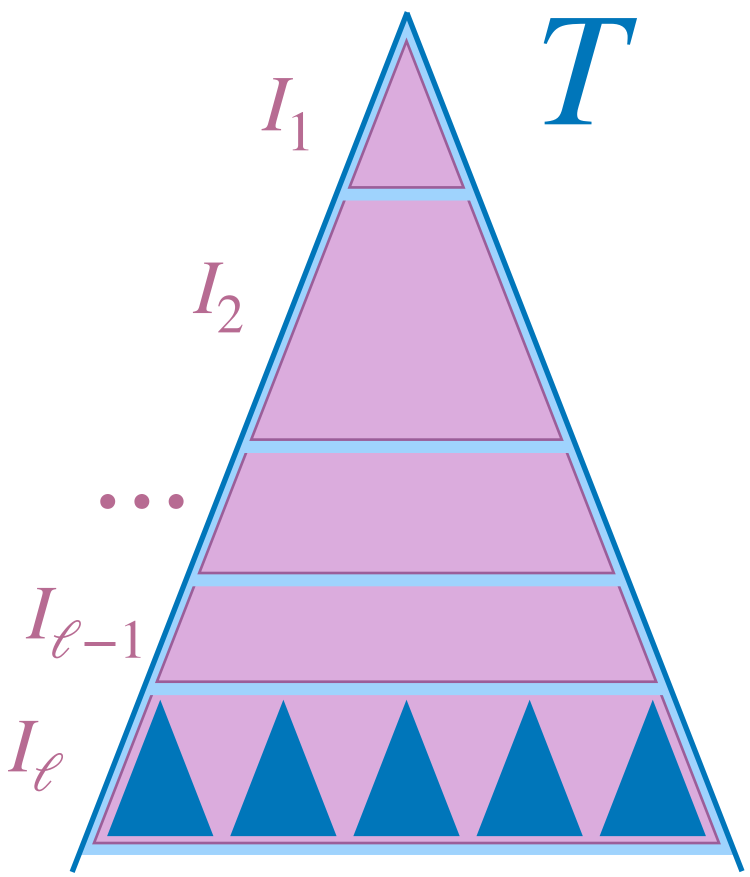
$$\sum_{i=1}^m \sqrt{|\mathscr{E}_i|} \le C^\ell \sqrt{\binom{d}{\ell}}\,.$$

$$\|\!| L_\ell T |\!\|\ \le (cC)^\ell \sqrt{\binom{d}{\ell}} \times \Lambda_{n^2,\ell}(\mathrm{dns}(T))$$

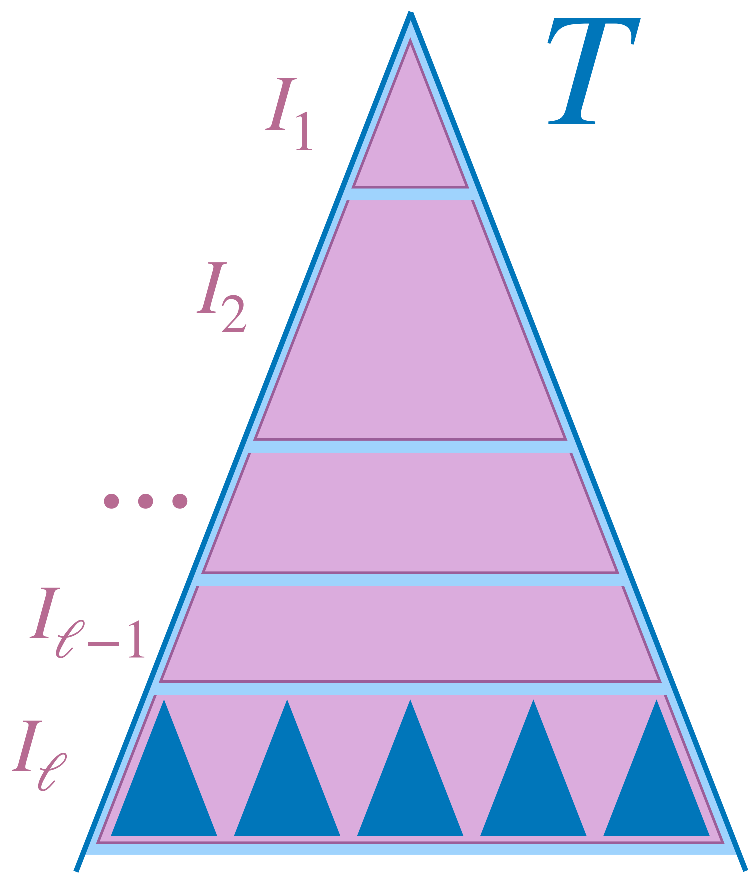
Fourier weight of decision trees

Analysis of the individual parts (Theorem 1.)

Analysis of the individual part



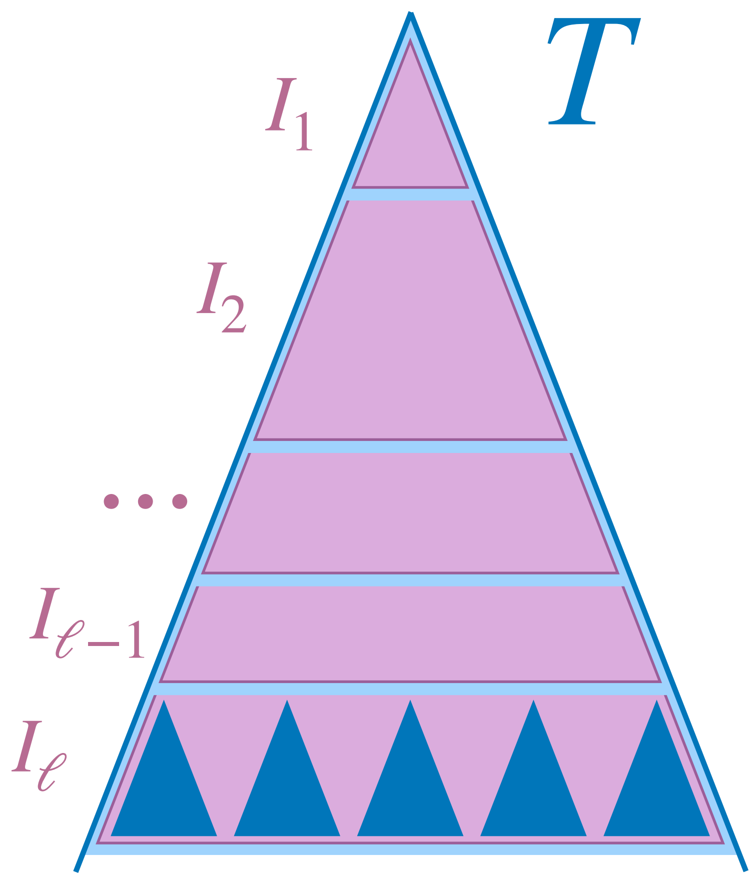
Analysis of the individual part



Theorem 1. For any elementary family $\mathscr{E} = I_1 * I_2 * \cdots * I_\ell$, we have

$$\left|\!\left|\!\left| \, T|_{\mathscr{E}} \right|\!\right|\!\right| \leq c_0^{\ell} c_1^{\ell-1} \sqrt{\prod_{i=1}^{\ell} |I_i| \, \Lambda_{n^2,\ell}(\mathrm{dns}(T))} \, .$$

Analysis of the individual part



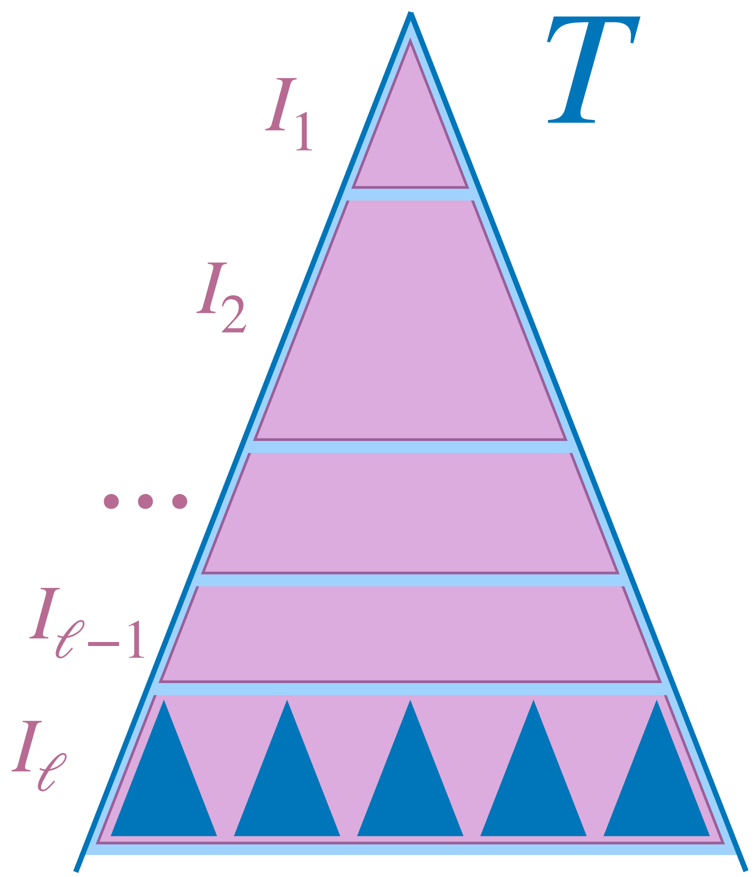
Theorem 1. For any elementary family $\mathscr{E} = I_1 * I_2 * \cdots * I_\ell$, we have

$$|||\, T|_{\mathscr{E}} \,||| \leq c_0^{\ell} c_1^{\ell-1} \sqrt{\prod_{i=1}^{\ell} |\, I_i \,|\, \Lambda_{n^2,\ell}(\mathrm{dns}(T))}\,.$$

Preview: Induction on ℓ.

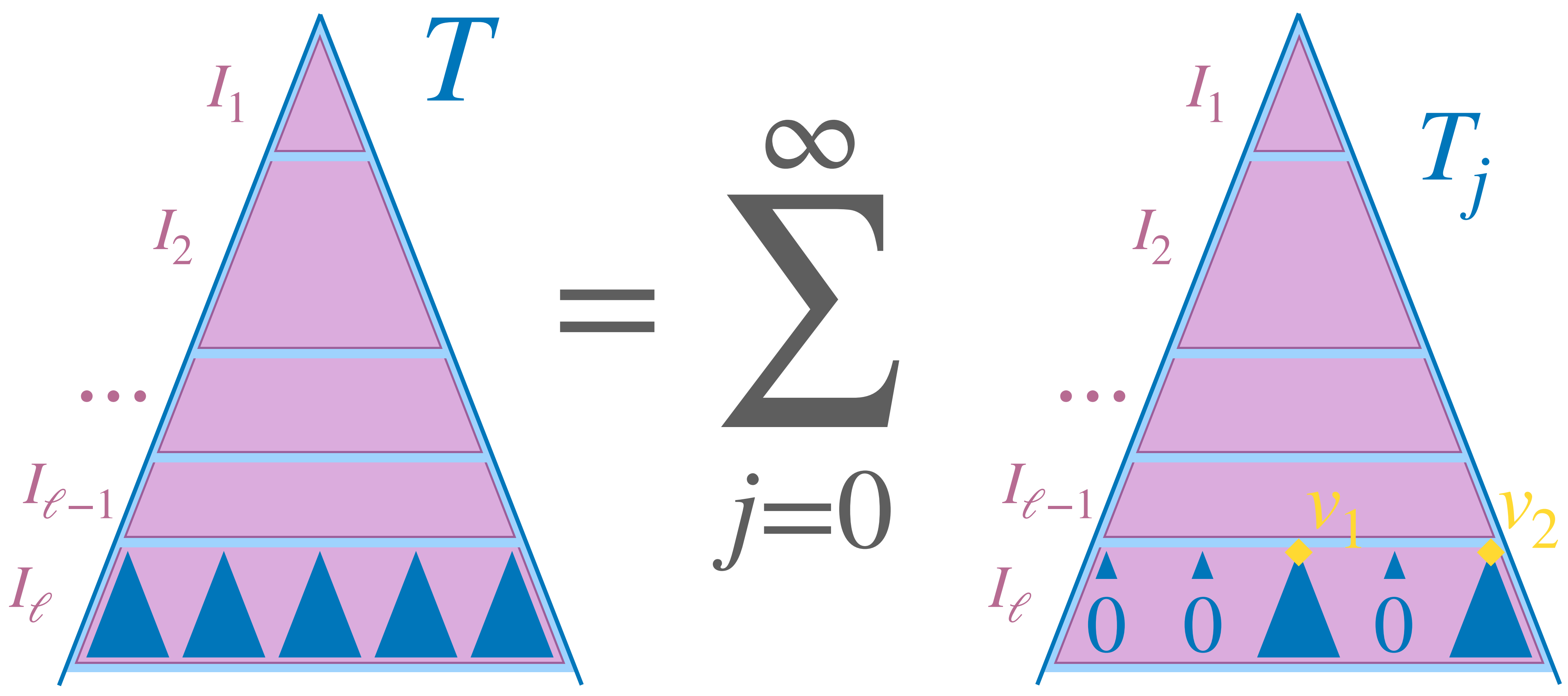
Analysis of the individual part

1. "Bucketing"



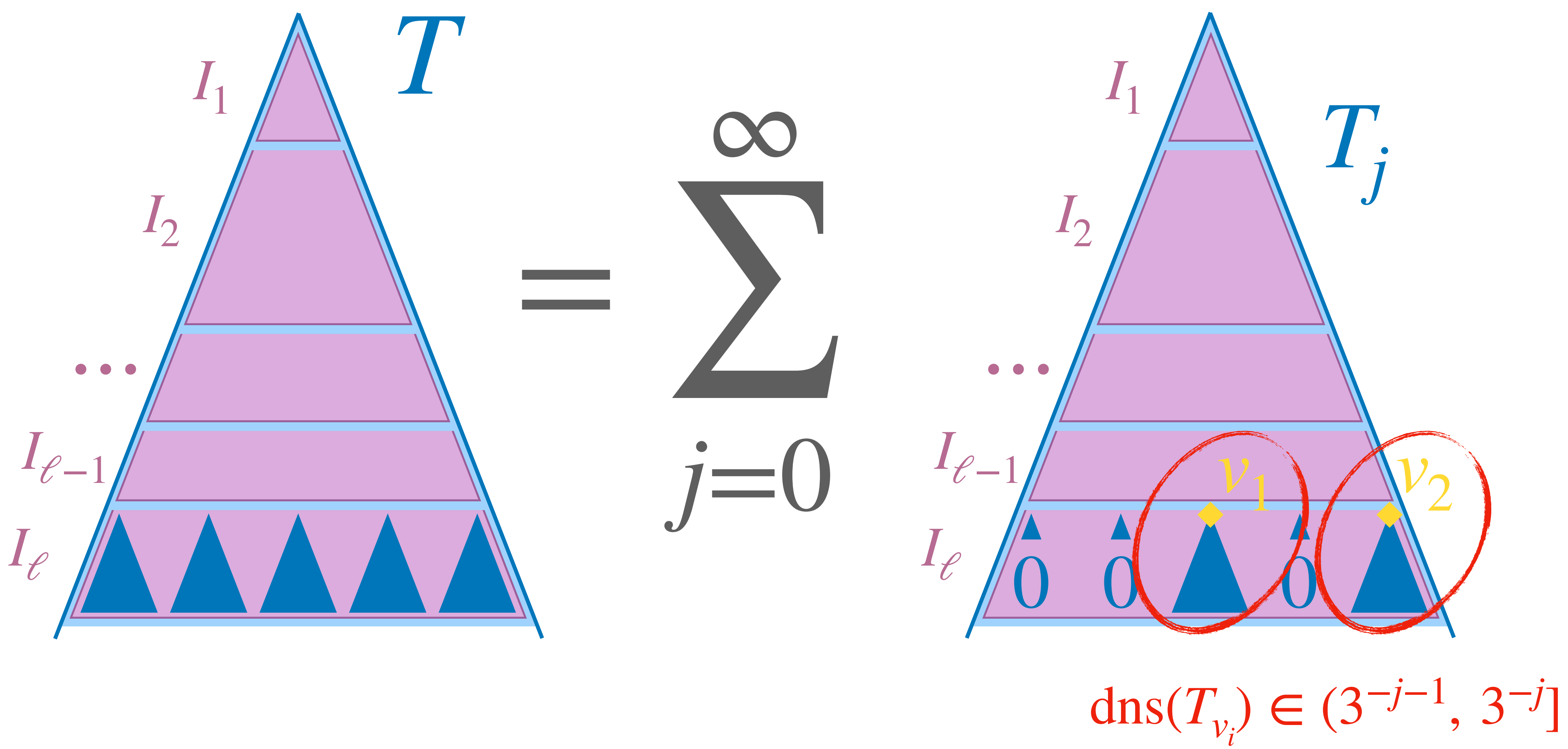
Analysis of the individual part

1. "Bucketing"



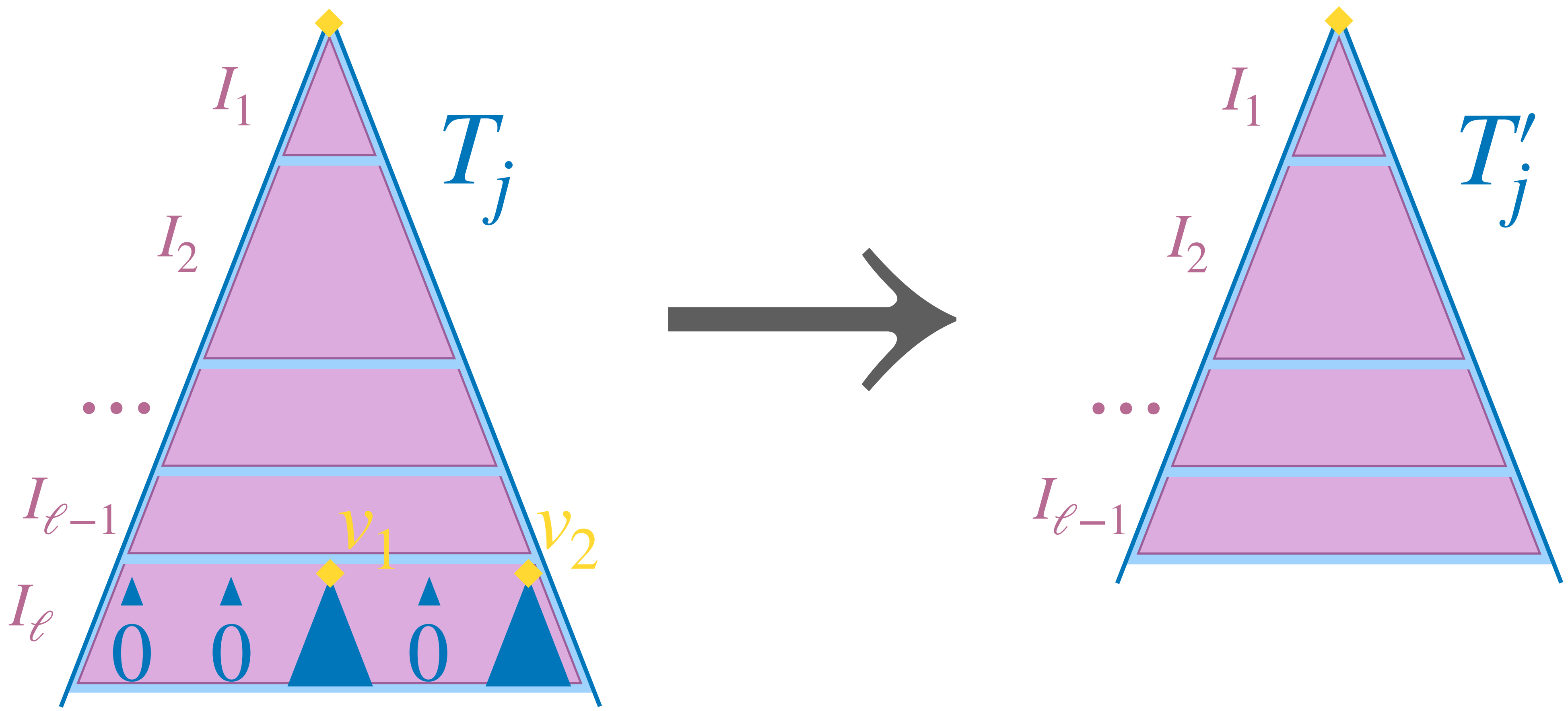
Analysis of the individual part

1. "Bucketing"



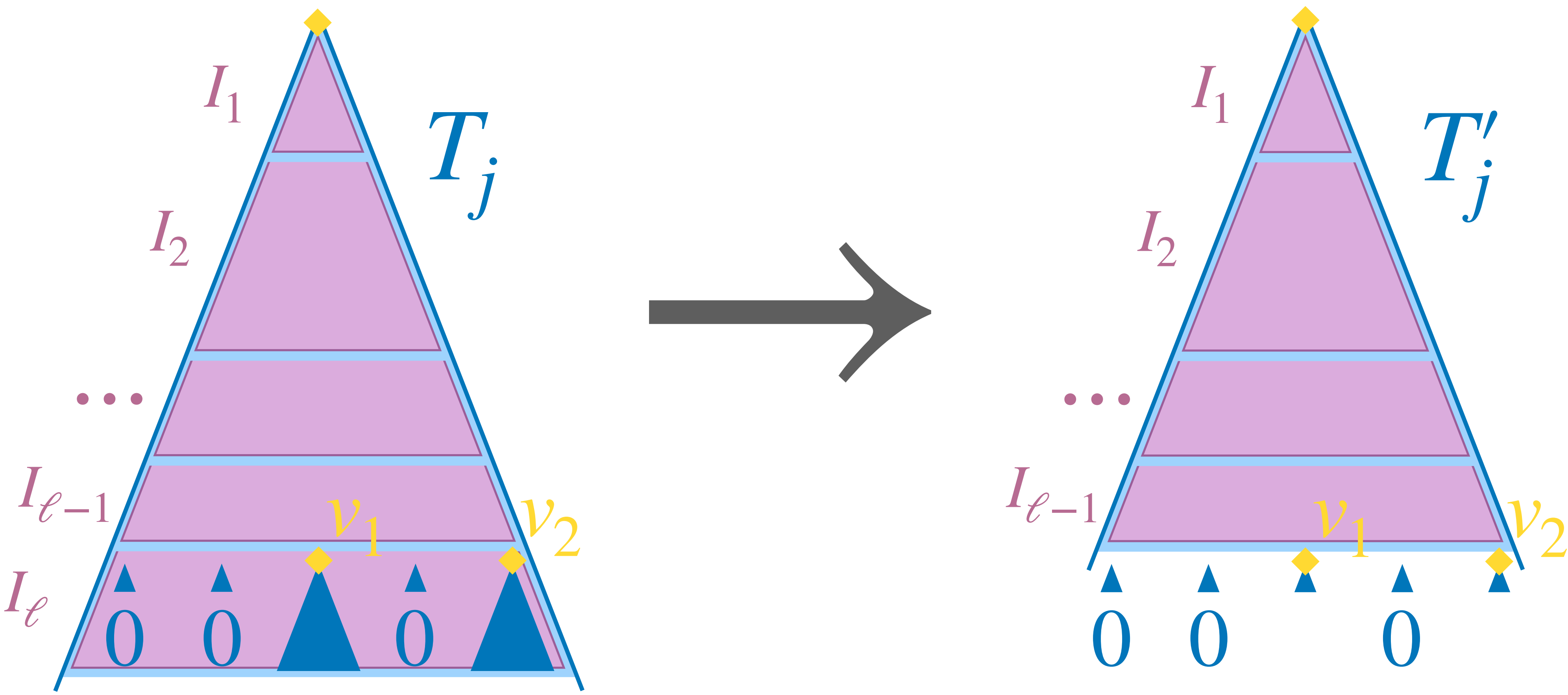
Analysis of the individual part

2. Replacing subtrees by their linear parts



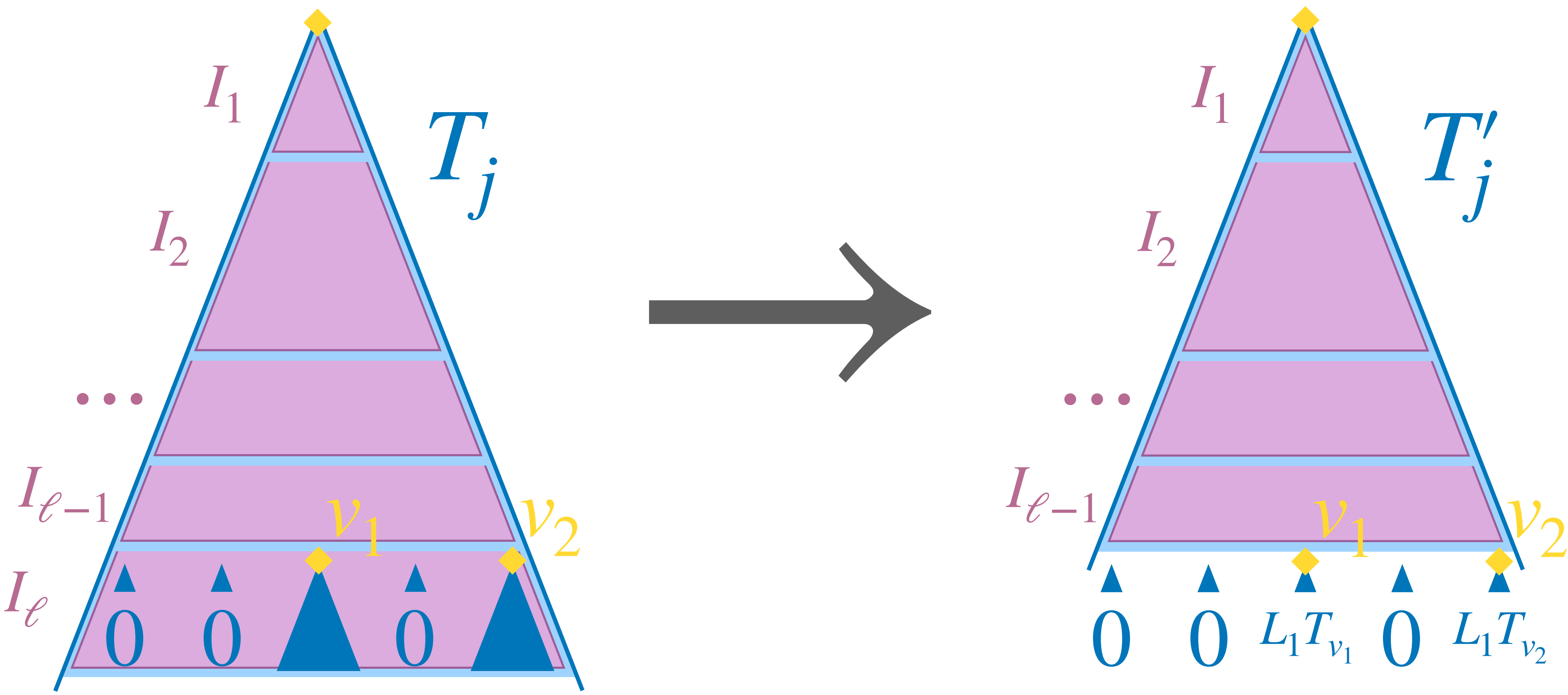
Analysis of the individual part

2. Replacing subtrees by their linear parts



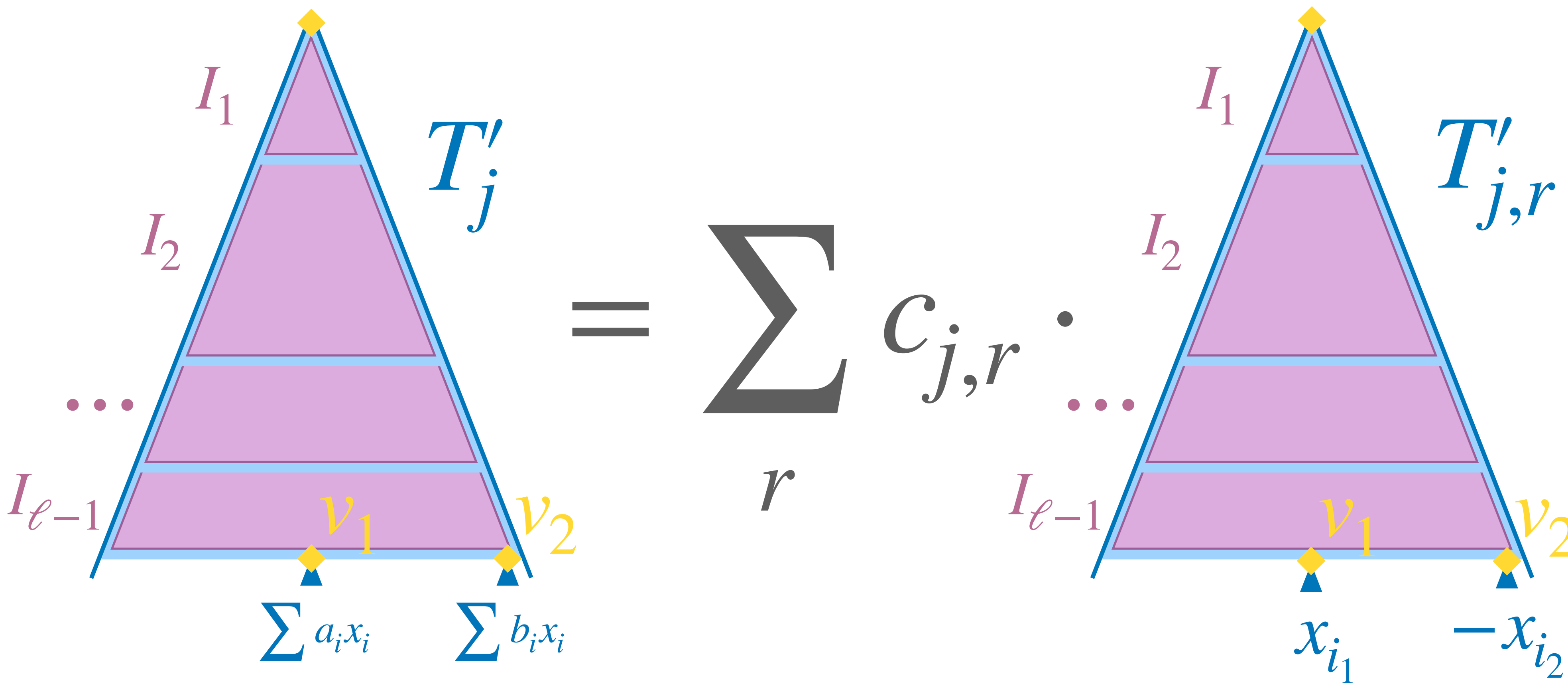
Analysis of the individual part

2. Replacing subtrees by their linear parts



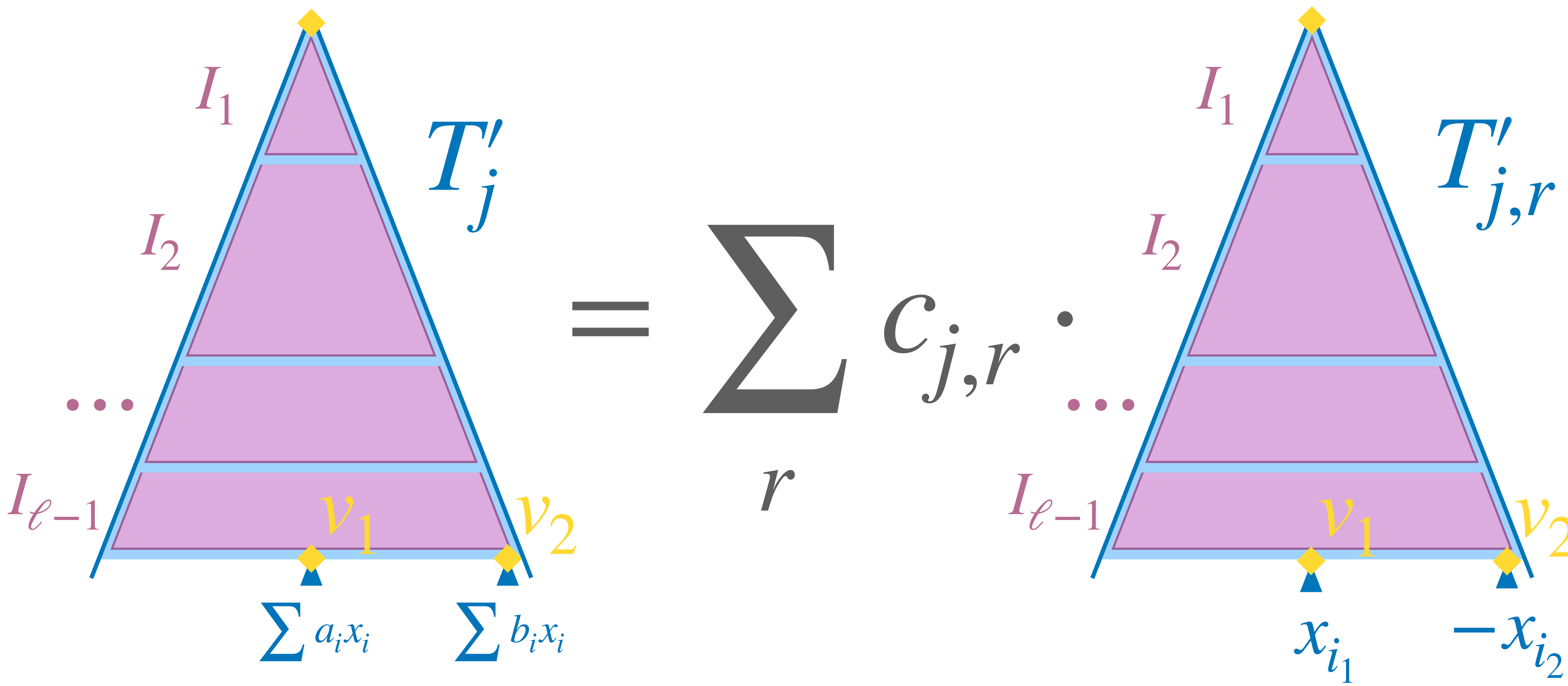
Analysis of the individual part

3. “Monomialization”



Analysis of the individual part

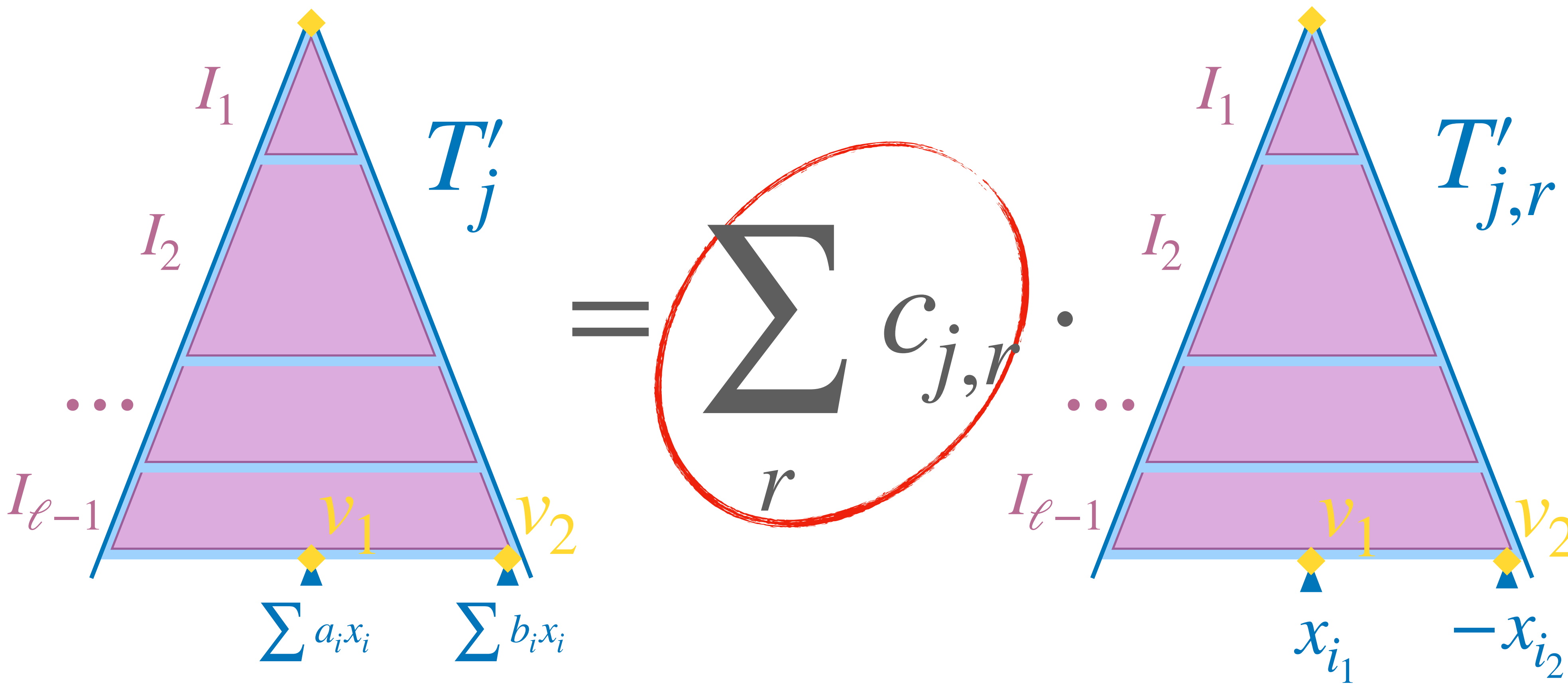
3. "Monomialization"



Make sure: $\mathrm{dns}(T'_{j,r}) \leq \mathrm{dns}(T'_j)$

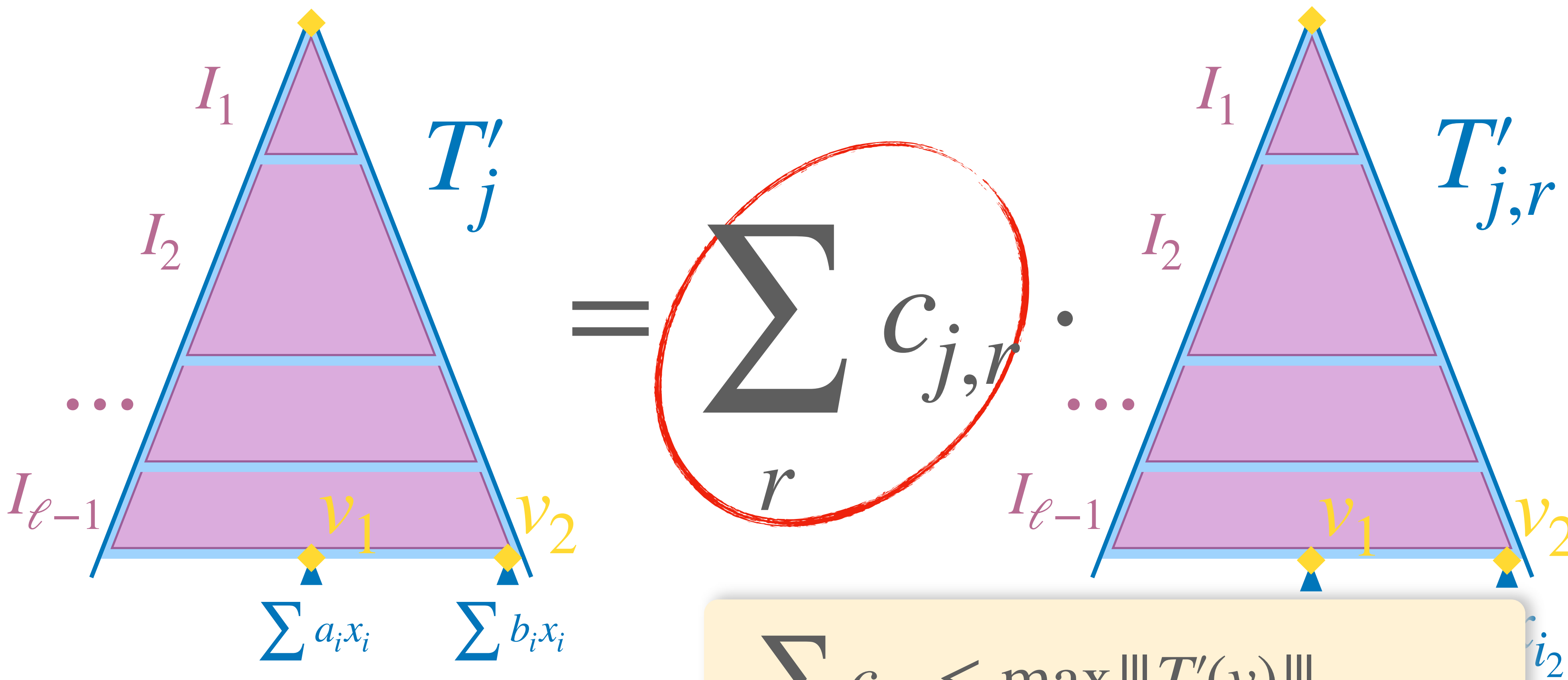
Analysis of the individual part

3. “Monomialization”



Analysis of the individual part

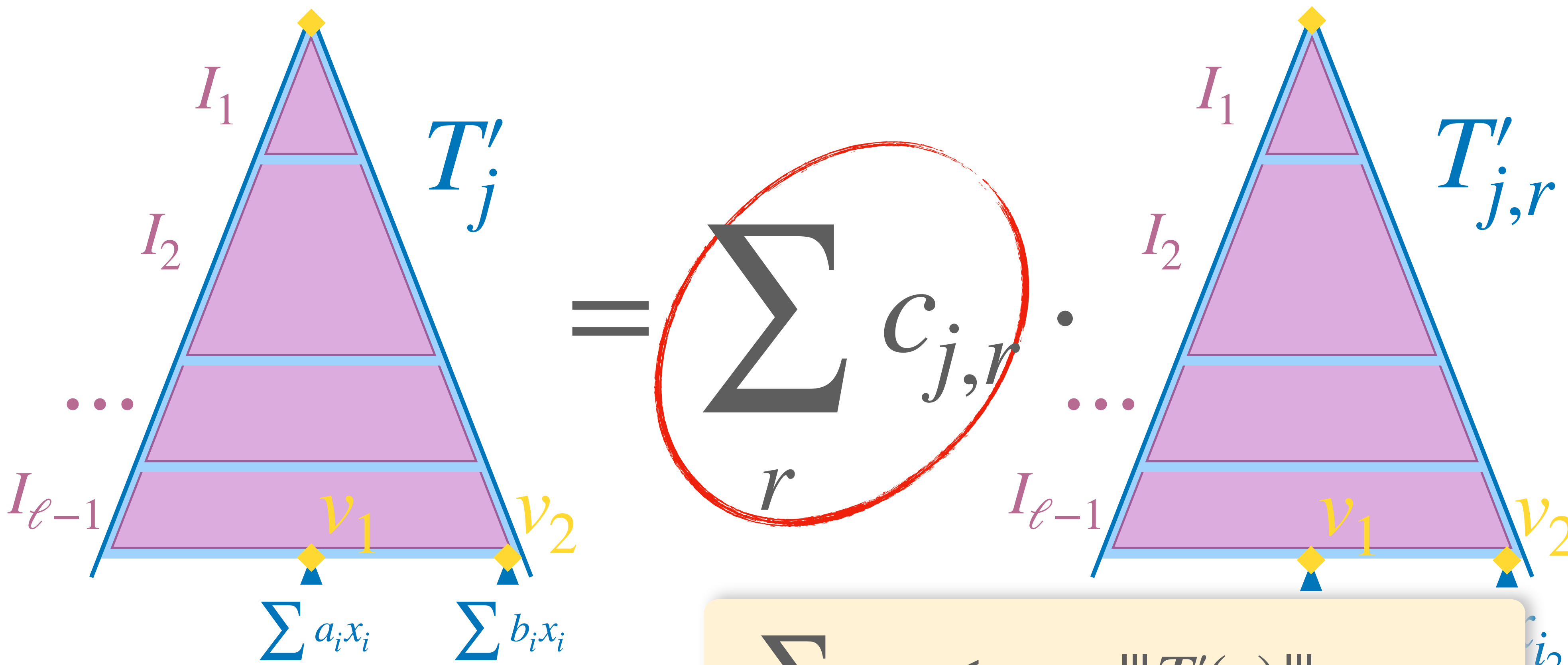
3. “Monomialization”



$$\sum_r c_{j,r} \leq \max_v |||T'_j(v)|||$$

Analysis of the individual part

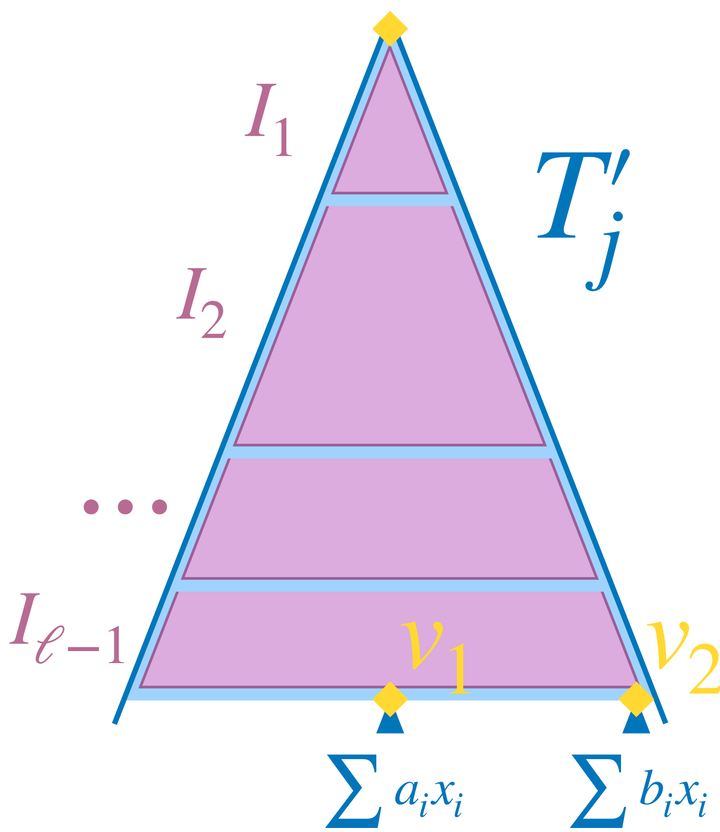
3. "Monomialization"



$$\sum_r c_{j,r} \le \max_v \|| T'_j(v) \|| \le c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j}) \,.$$

Analysis of the individual part

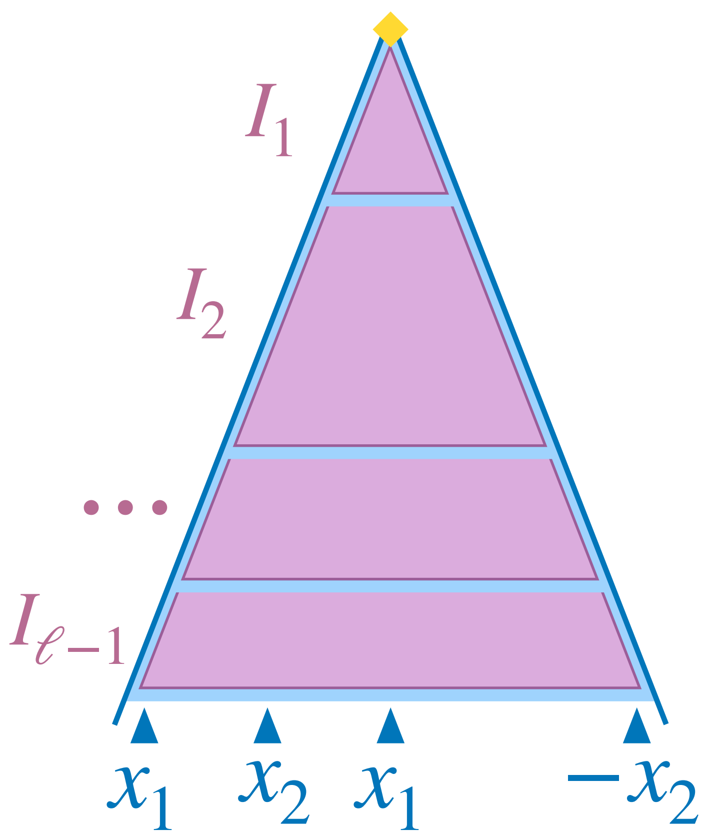
3. "Monomialization"



$$\in c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j}) \times \mathrm{conv}\{\mathcal{T}_{\mathrm{dns}(T_j)}\}$$

Analysis of the individual part

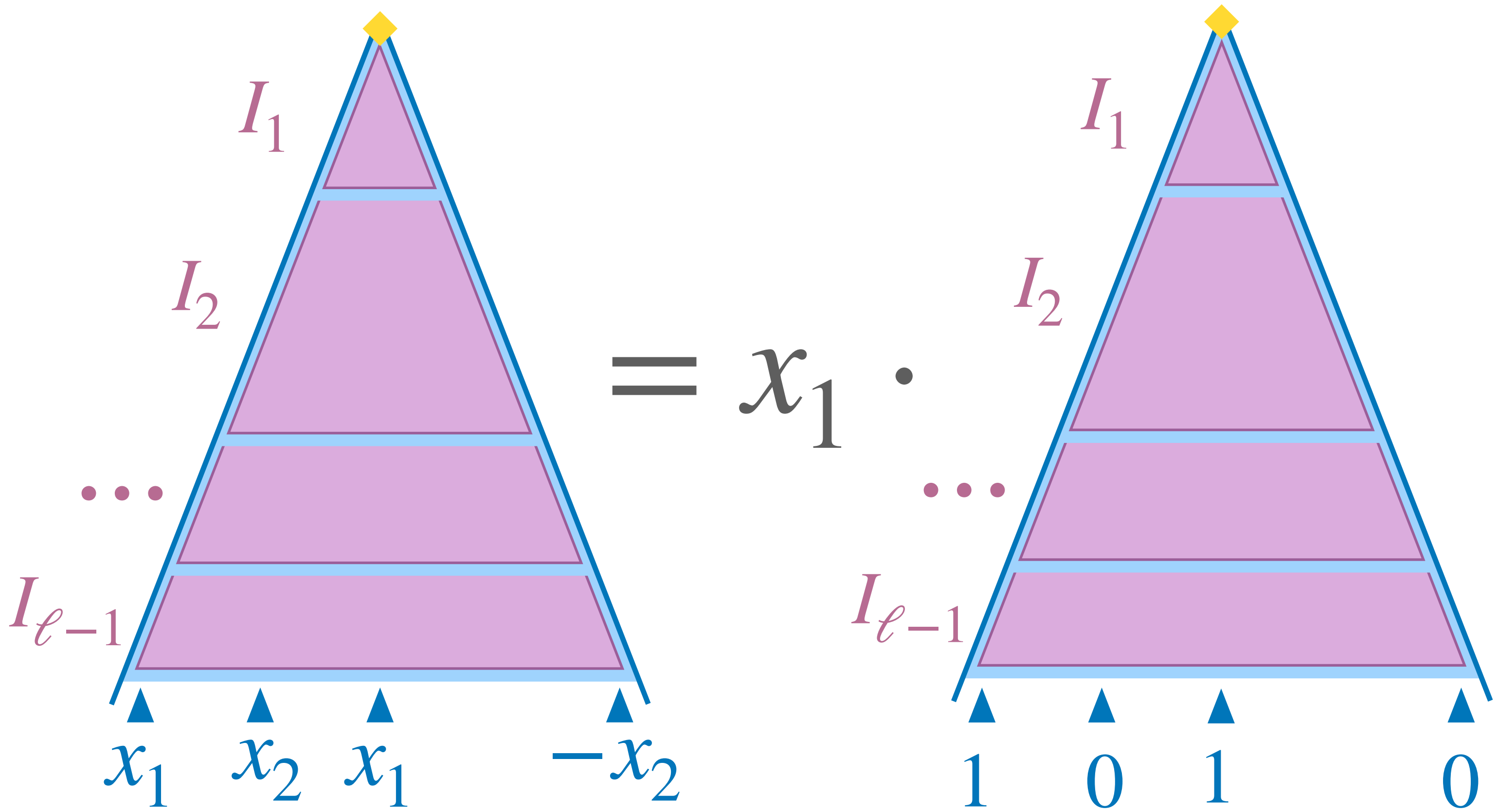
4. Grouping by variable



$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

Analysis of the individual part

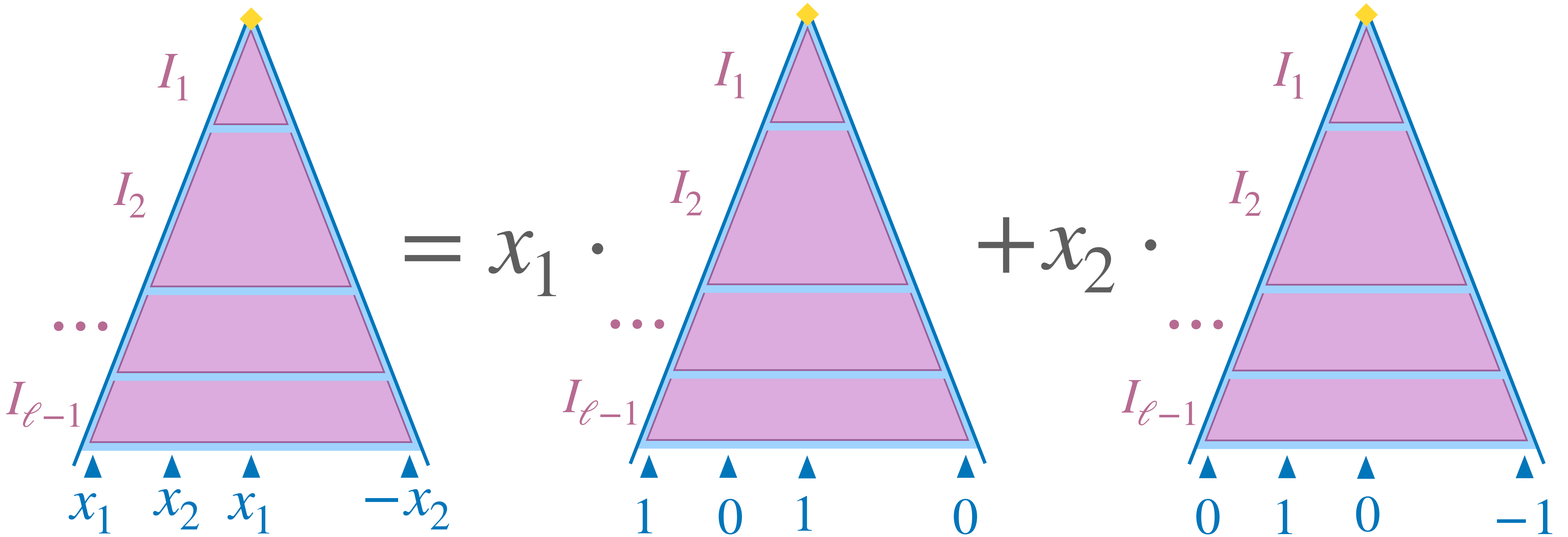
4. Grouping by variable



$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

Analysis of the individual part

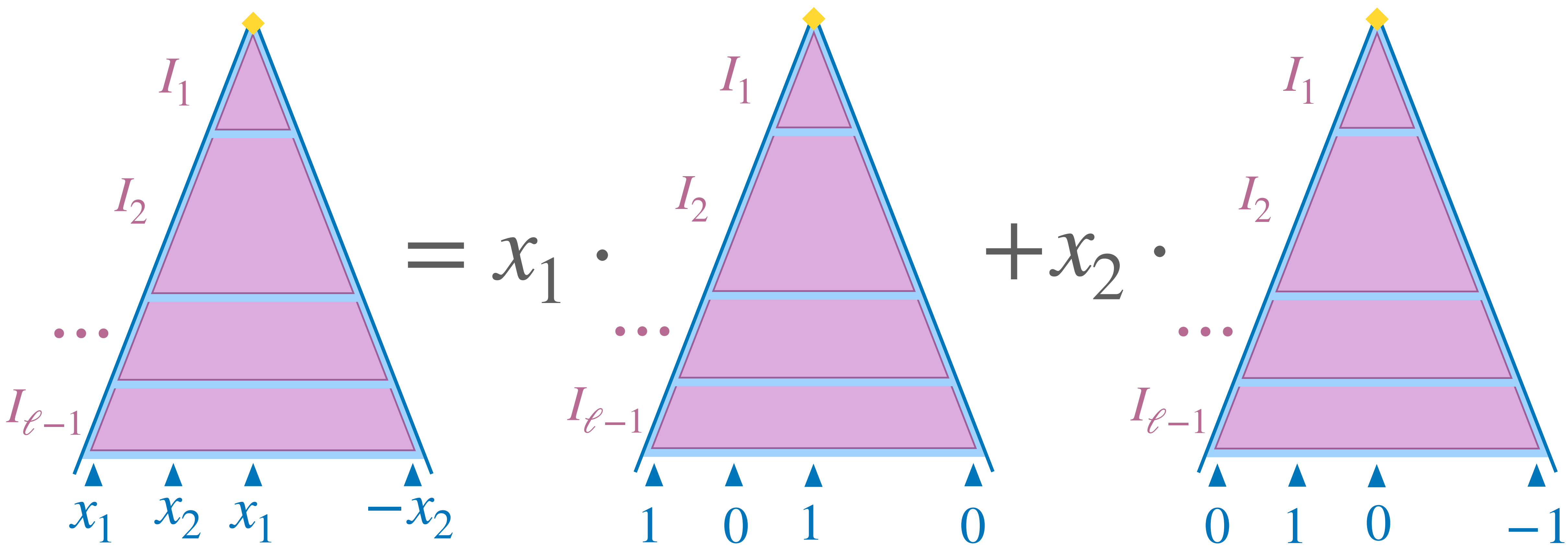
4. Grouping by variable



$T'_{j,r} \in \mathcal{T}_{\mathrm{dns}(T_j)}$

Analysis of the individual part

4. Grouping by variable

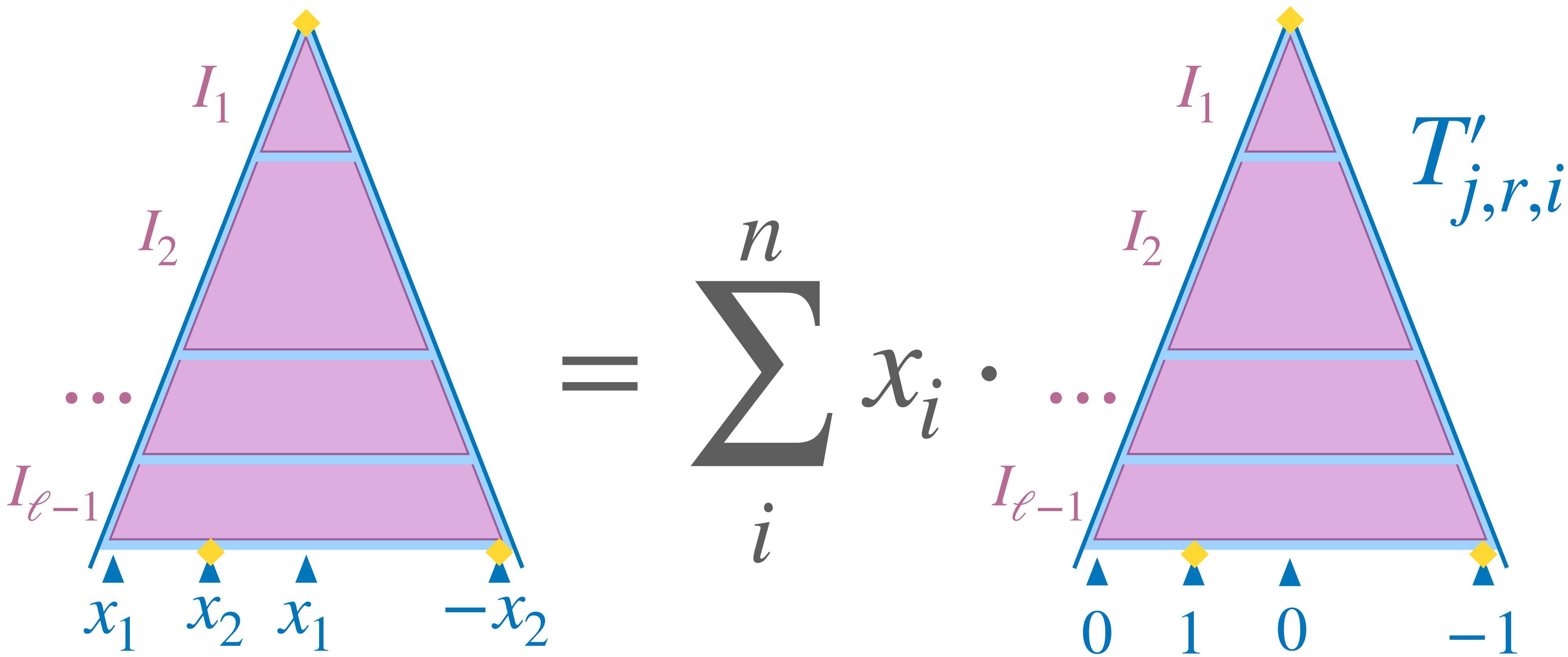


$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

$+ \cdots$

Analysis of the individual part

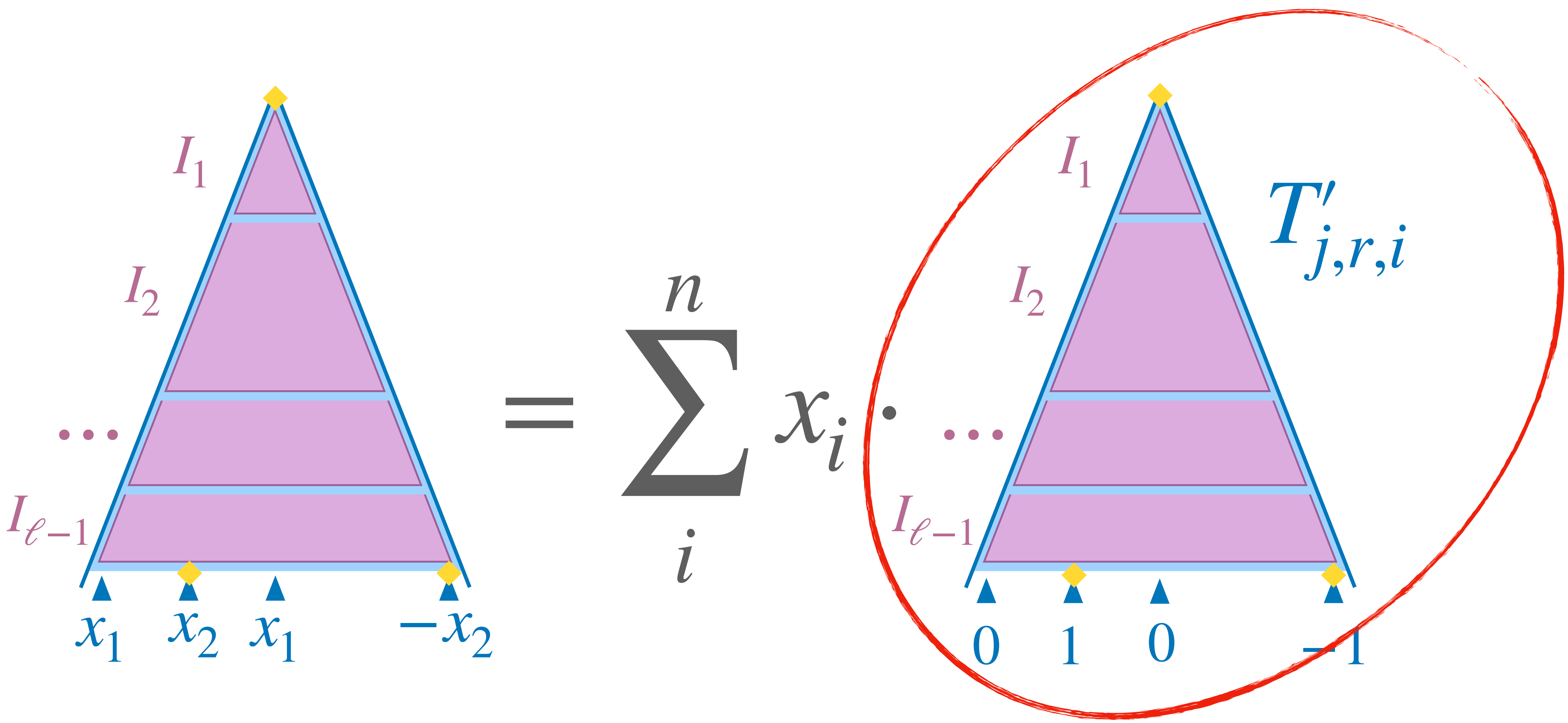
4. Grouping by variable



$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

Analysis of the individual part

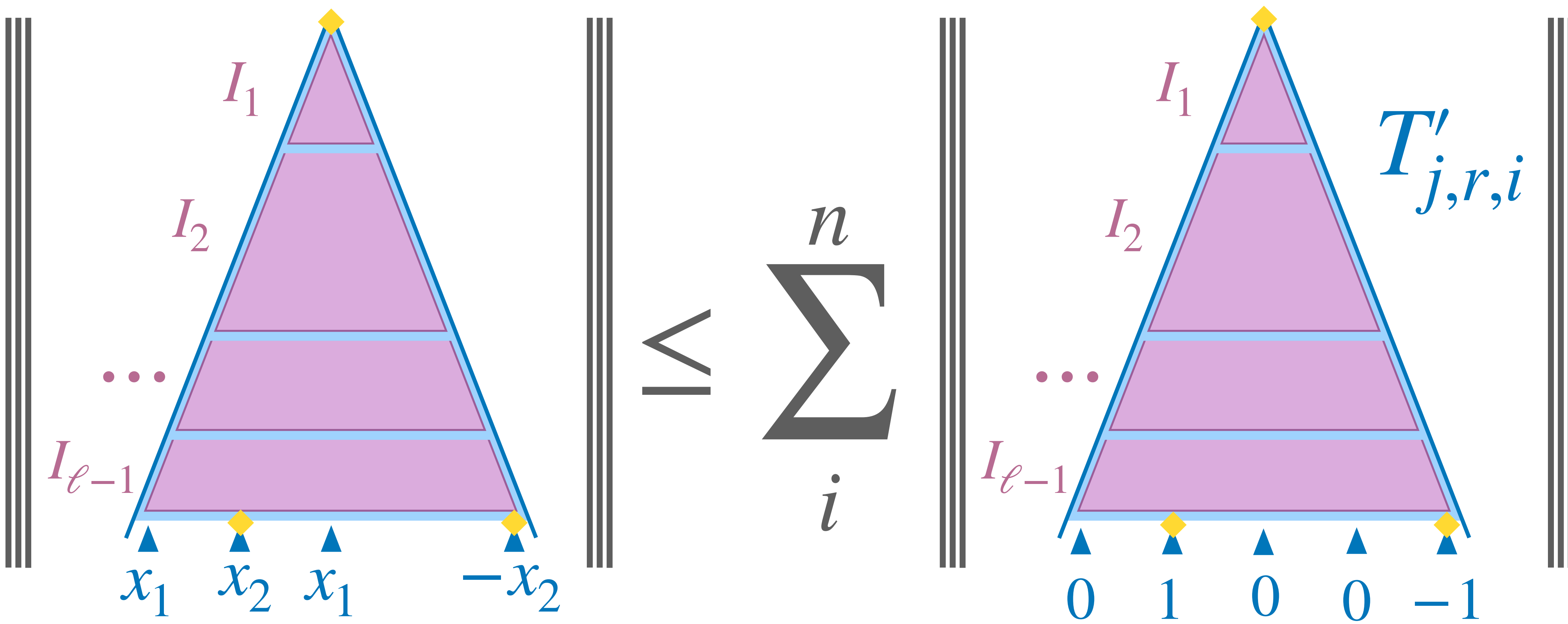
4. Grouping by variable



$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

Analysis of the individual part

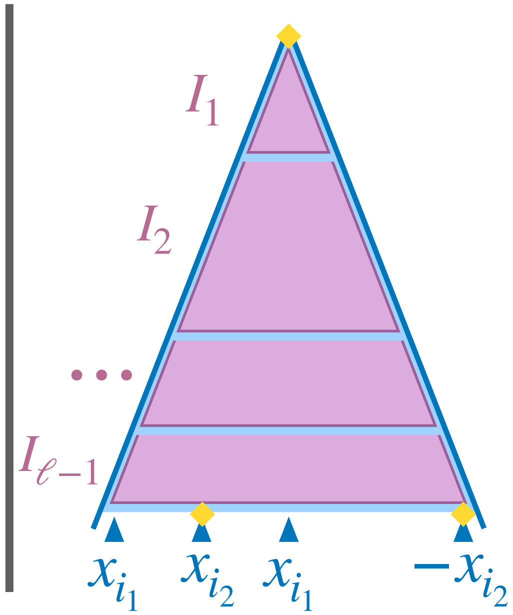
4. Grouping by variable



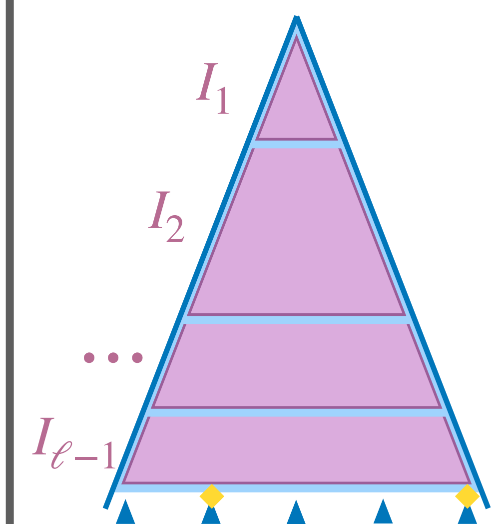
$T'_{j,r} \in \mathcal{T}_{\mathrm{dns}(T_j)}$

Analysis of the individual part

4. Grouping by variable



$$\leq \sum_{i}^{n}$$

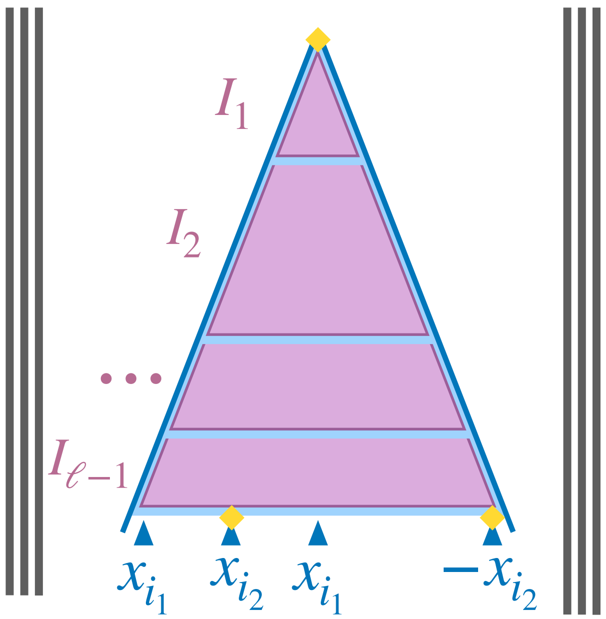


$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

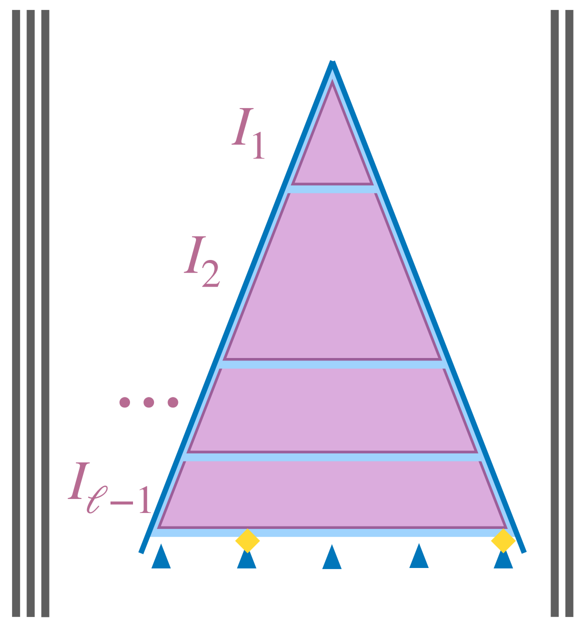
$$\leq \sum_{i=1}^{n} c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s| \, \Lambda_{n^2,\ell-1}(\mathrm{dns}(T'_{j,r,i}))}$$

Analysis of the individual part

4. Grouping by variable



$\leq \sum_{i}^{n}$



$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

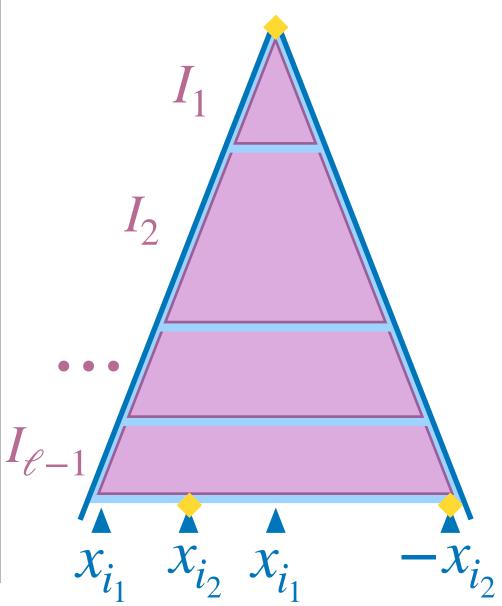
$$\leq \sum_{i=1}^{n} c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s| \, \Lambda_{n^2,\ell-1}(\mathrm{dns}(T'_{j,r,i}))}$$

(concavity)

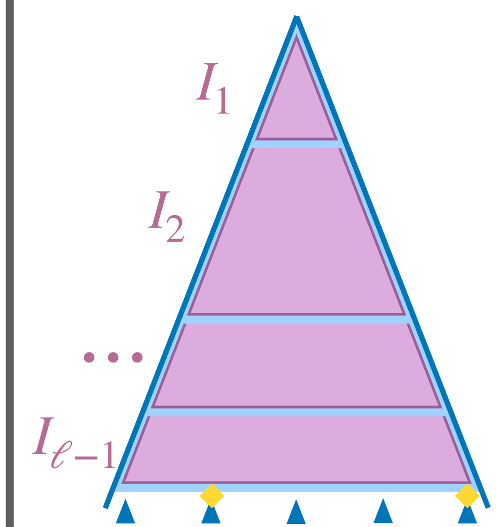
$$\leq c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s| \, n \Lambda_{n^2,\ell-1}\left(\frac{\sum_{i=1}^{n} \mathrm{dns}(T'_{j,r,i})}{n}\right)}$$

Analysis of the individual part

4. Grouping by variable



$$\leq \sum_{i}^{n}$$



$T'_{j,r} \in \mathscr{T}_{\mathrm{dns}(T_j)}$

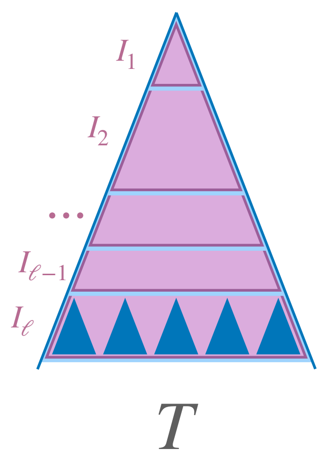
$$\leq \sum_{i=1}^{n} c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s| \, \Lambda_{n^2,\ell-1}(\mathrm{dns}(T'_{j,r,i}))}$$

(concavity)

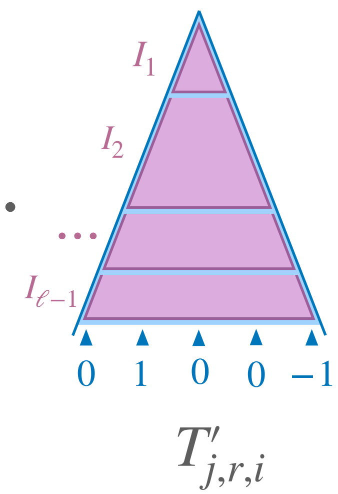
$$\leq c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s| \, n \Lambda_{n^2,\ell-1}\left(\frac{\mathrm{dns}(T_j)}{n}\right)}$$

Analysis of the individual part

5. Putting things together



$$= \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n} x_i \cdot$$



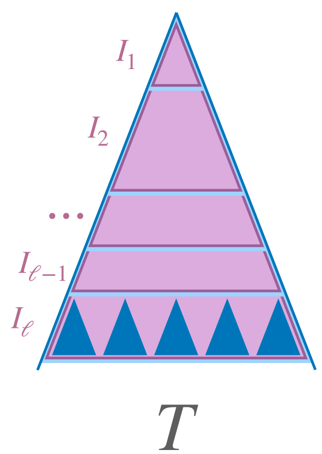
Bucketing

Monomialization

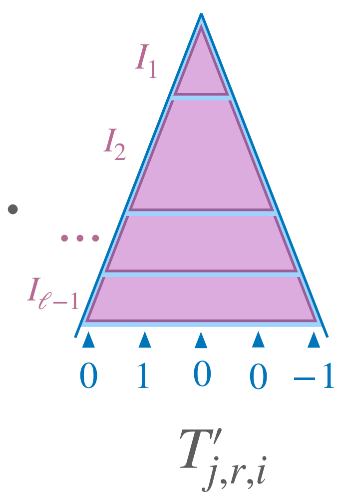
Group by variable

Analysis of the individual part

5. Putting things together



$$= \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n} x_i \cdot$$



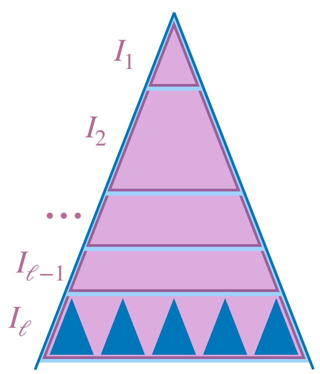
Bucketing

$$\in c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j})$$

$$\times \, \mathrm{conv}\{\mathscr{T}_{\mathrm{dns}(T_j)}\}$$

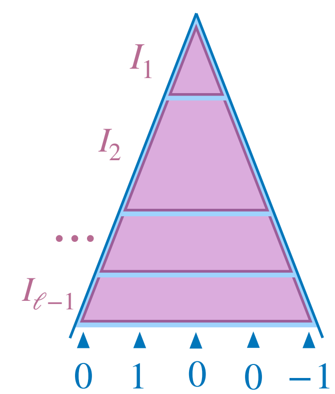
Analysis of the individual part

5. Putting things together

$\Vert$  $\Vert$

T

$$\le \sum_{j=0}^{\infty} \left\Vert \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n} \right.$$



$\Vert$

$T'_{j,r,i}$

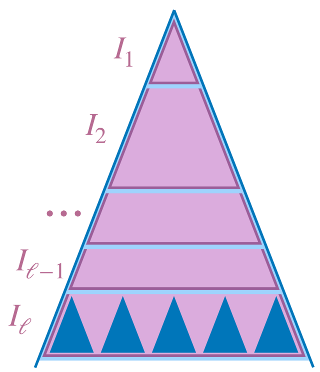
Bucketing

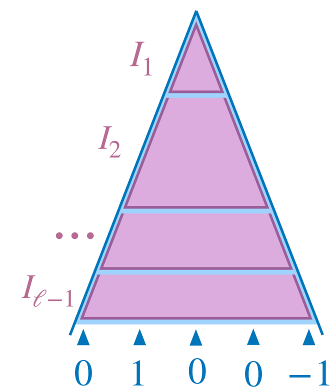
$$\in c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j})$$

$$\times \operatorname{conv}\{\mathscr{T}_{\mathrm{dns(T_j)}}\}$$

Analysis of the individual part

5. Putting things together



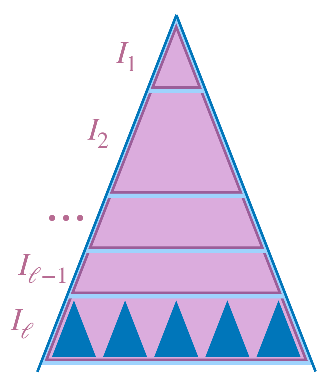


$$\left|\!\left|\!\left| T \right|\!\right|\!\right| \leq \sum_{j=0}^{\infty} \left\| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n} T'_{j,r,i} \right\|$$

$$\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \mathrm{conv}\{\mathcal{T}_{\mathrm{dns}(T_j)}\}} \left|\!\left|\!\left| T \right|\!\right|\!\right|$$

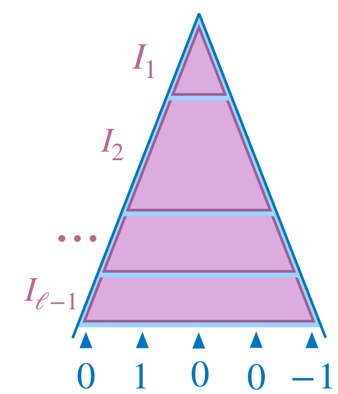
Analysis of the individual part

5. Putting things together

$\vert\vert\vert$  $\vert\vert\vert$

T

$$\le \sum_{j=0}^{\infty} \left|\!\left|\!\left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n} \right.\right.\right.$$

 $\vert\vert\vert$

$T'_{j,r,i}$

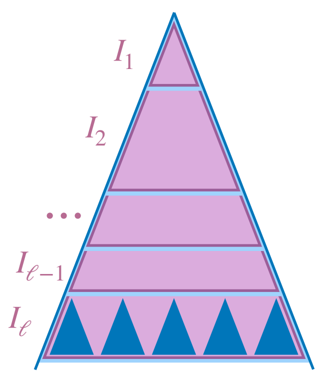
$$\le \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \operatorname{conv}\{\mathscr{T}_{\operatorname{dns}(T_j)}\}} |||T|||$$

$$\le c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s|\, n \Lambda_{n^2,\ell-1}\left(\frac{\operatorname{dns}(T_j)}{n}\right)}$$

Analysis of the individual part

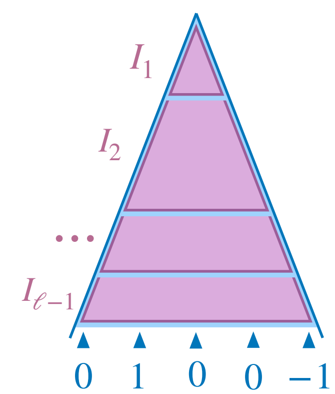
5. Putting things together

$$\left|\!\left|\!\left| \vphantom{\sum} \right.\right.\right.$$



T

$$\left.\left.\left. \vphantom{\sum} \right|\!\right|\!\right| \le \sum_{j=0}^{\infty} \left|\!\left|\!\left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n} \right.\right.\right.$$



$T'_{j,r,i}$

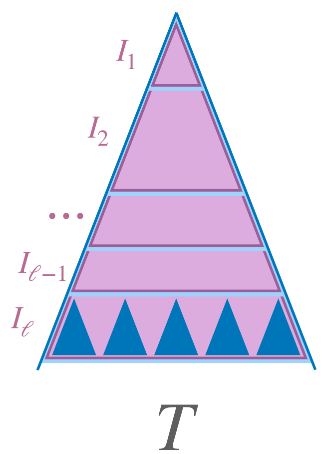
$$\left.\left.\left. \vphantom{\sum} \right|\!\right|\!\right|$$

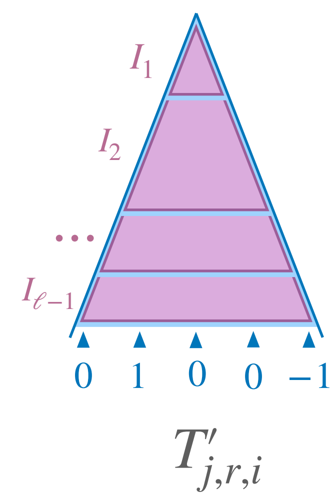
$$\le \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \mathrm{conv}\{\mathscr{T}_{\mathrm{dns(T_j)}}\}} |||T|||$$

$$\le c_0^{\ell} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell} |I_s|}\, \sum_{j=0}^{\infty} \Lambda_{n^2,1}(3^{-j})\, n \Lambda_{n^2,\ell-1}\left(\frac{\mathrm{dns}(T_j)}{n}\right)$$

Analysis of the individual part

5. Putting things together

$\Bigg|\Bigg|\Bigg|$ 

$\Bigg|\Bigg|\Bigg| \leq \sum_{j=0}^{\infty} \Bigg|\Bigg|\Bigg| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n}$ 

$\Bigg|\Bigg|\Bigg|$

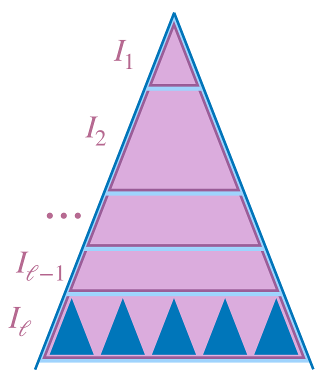
$$\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \mathrm{conv}\{\mathcal{T}_{\mathrm{dns(T_j)}}\}} |||T|||$$

$$\leq c_0^\ell c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell} |I_s|} \sum_{j=0}^{\infty} \Lambda_{n^2,1}(3^{-j}) n \Lambda_{n^2,\ell-1}\left(\frac{\mathrm{dns}(T_j)}{n}\right)$$

$\leq c_1 \Lambda_{n^2,\ell}(\mathrm{dns}(T))$

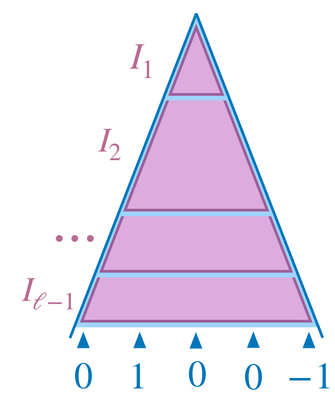
Analysis of the individual part

5. Putting things together



T

$$\left|\!\left|\!\left| \; T \; \right|\!\right|\!\right| \leq \sum_{j=0}^{\infty} \left|\!\left|\!\left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^{n} T'_{j,r,i} \right|\!\right|\!\right|$$



$T'_{j,r,i}$

$$\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|}\, \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \mathrm{conv}\{\mathscr{T}_{\mathrm{dns(T_j)}}\}} |||T|||$$

$$\leq c_0^{\ell} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell} |I_s|} \sum_{j=0}^{\infty} \Lambda_{n^2,1}(3^{-j})\, n \Lambda_{n^2,\ell-1}\left(\frac{\mathrm{dns}(T_j)}{n}\right)$$

$$\leq c_1 \Lambda_{n^2,\ell}(\mathrm{dns}(T))$$

Open problems

Problem 1

For total functions f, does one have $R_{1/3}(f) \leq O(Q_{1/3}(f)^3)$?

Problem 2

Are randomized and quantum communication complexity polynomially related for total functions?

Open problems

Problem 1

For total functions f, does one have $R_{1/3}(f) \leq O(Q_{1/3}(f)^3)$?

Problem 2

Are randomized and quantum communication complexity polynomially related for total functions?

Thank you!