

AN OPTIMAL SEPARATION OF RANDOMIZED AND QUANTUM QUERY COMPLEXITY

Alexander Sherstov, Andrey Storozhenko, and Pei Wu

UCLA

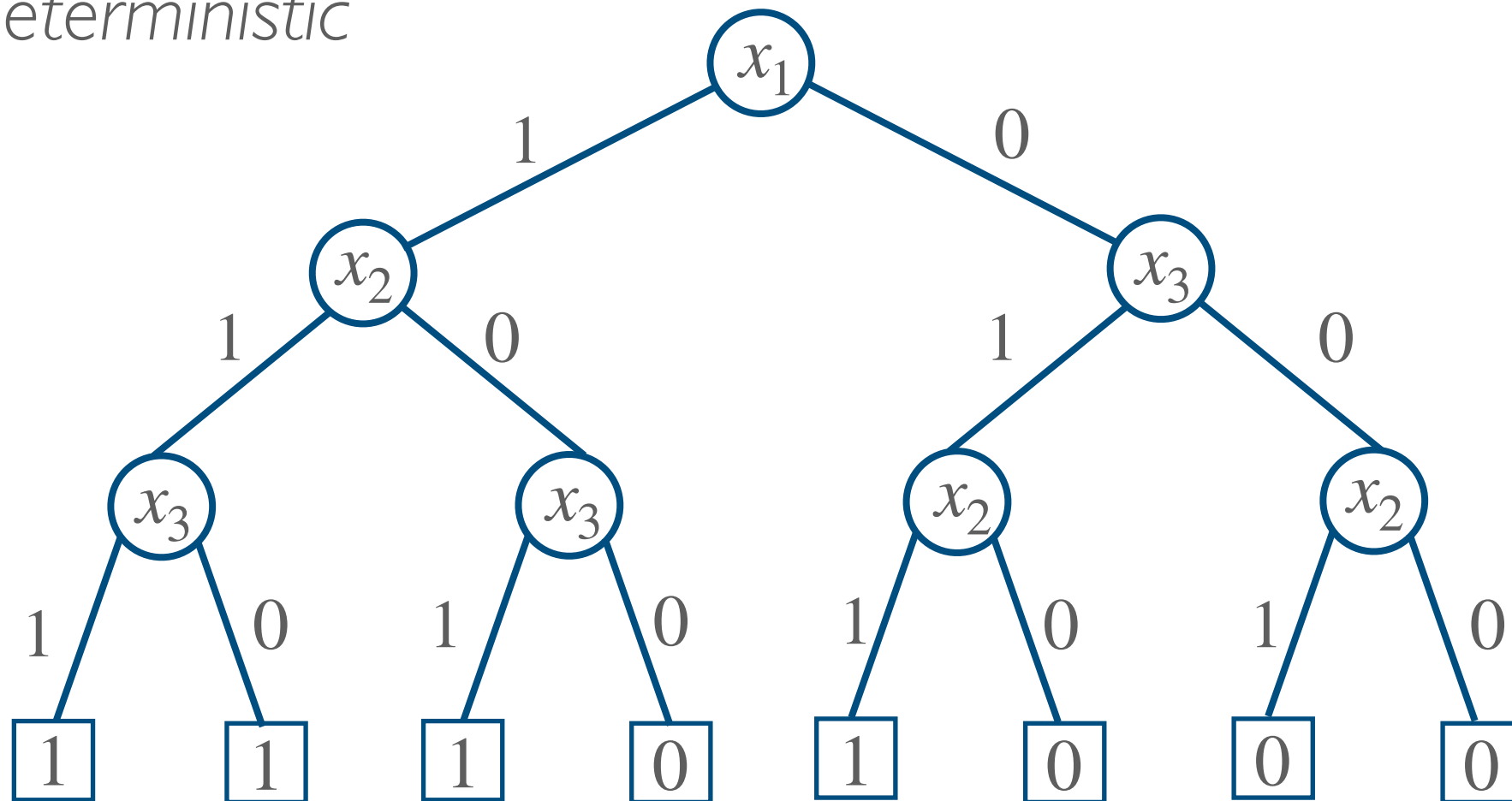
Central open problem

How much faster can quantum computers be than classical?

Most research focuses on the query model.

Query complexity

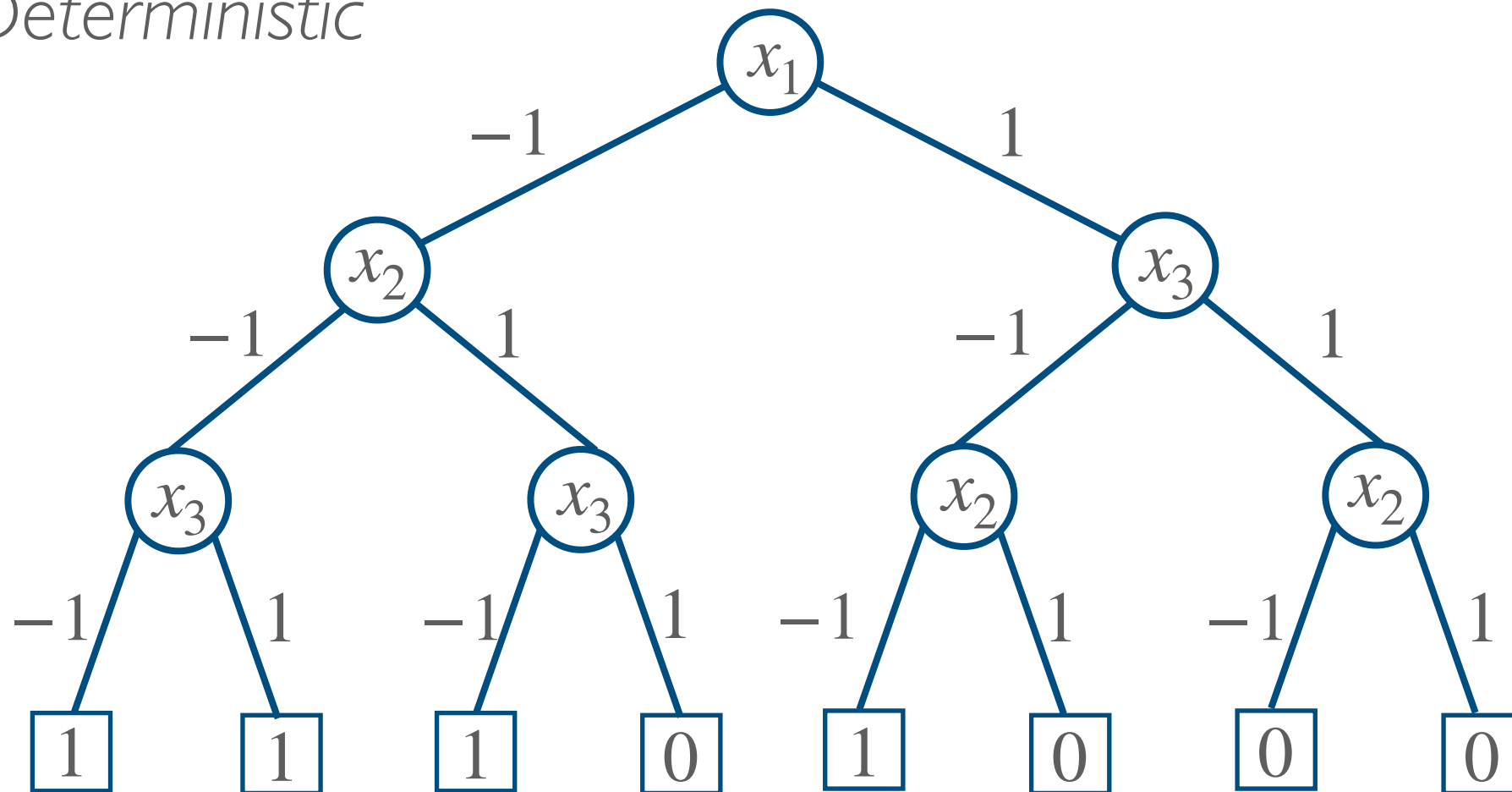
Deterministic



$$T : \{0,1\}^n \rightarrow \{0,1\}$$

Query complexity

Deterministic

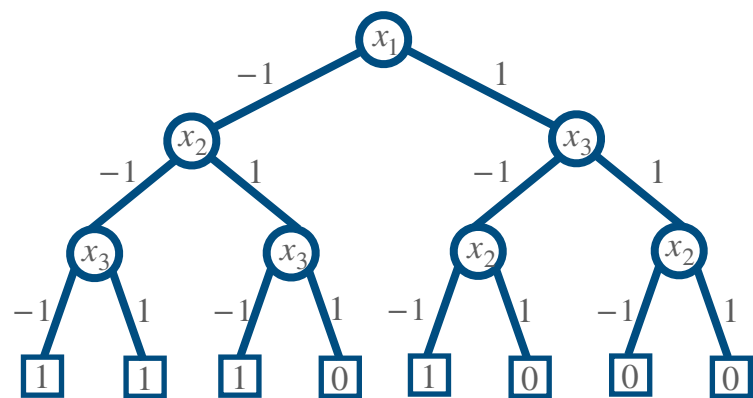


$$T : \{-1, 1\}^n \rightarrow \{0, 1\}$$

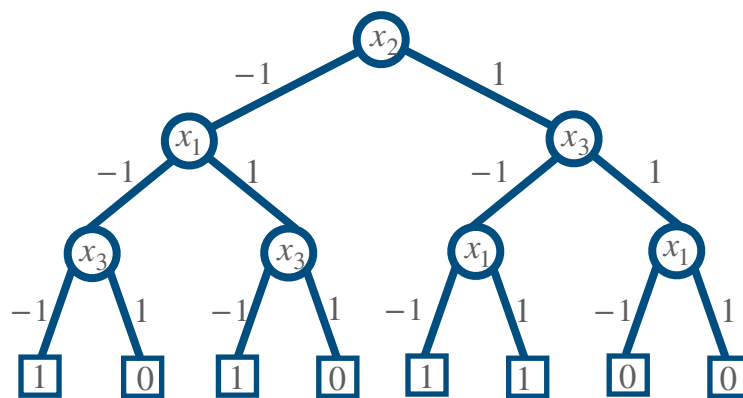
Query complexity



Randomized



T_1



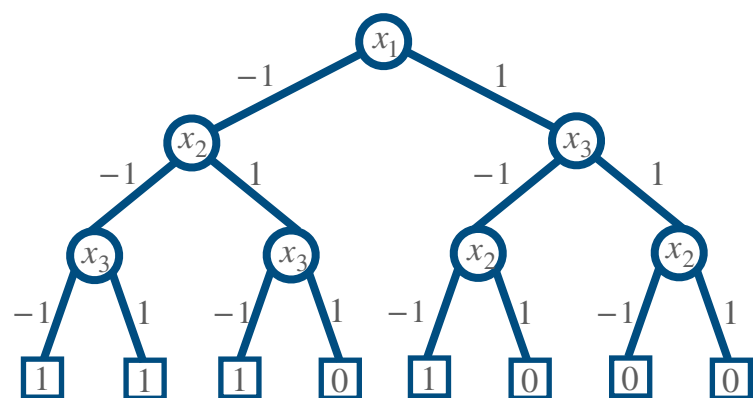
T_2

...

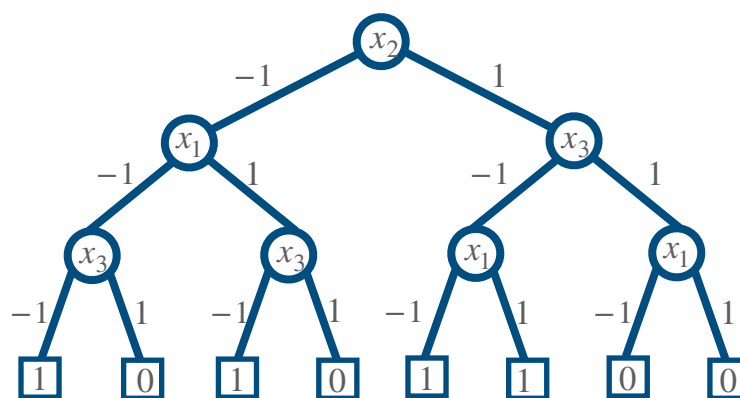
Query complexity



Randomized



T_1



T_2

...

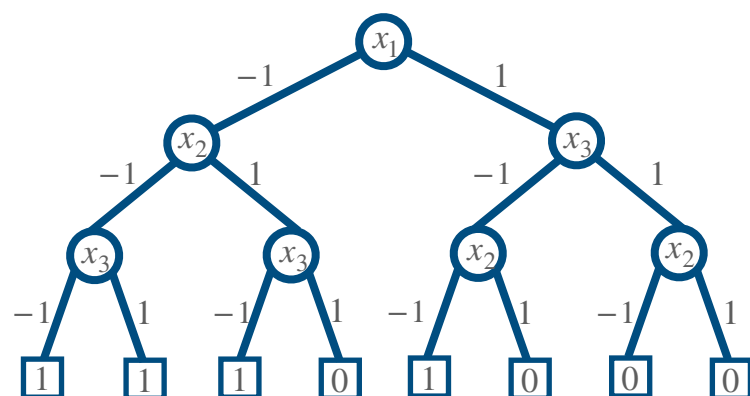
T computes $f : \{-1, 1\}^n \rightarrow \{0, 1\}$ with error ϵ if

$$\mathbf{P}_r[T_r(x) \neq f(x)] \leq \epsilon, \quad \forall x \in \{-1, 1\}^n.$$

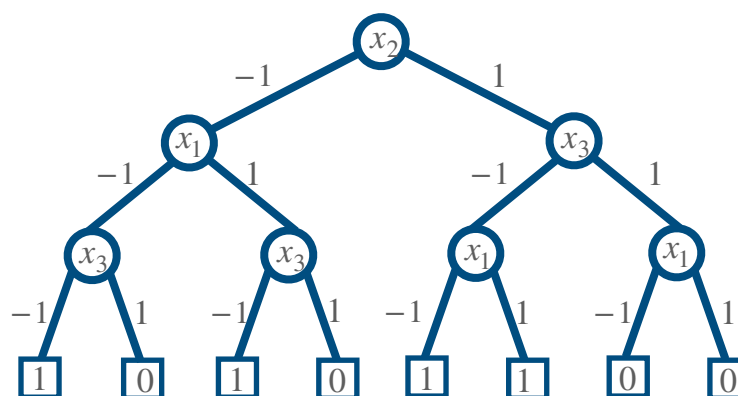
Query complexity



Randomized



T_1



T_2

...

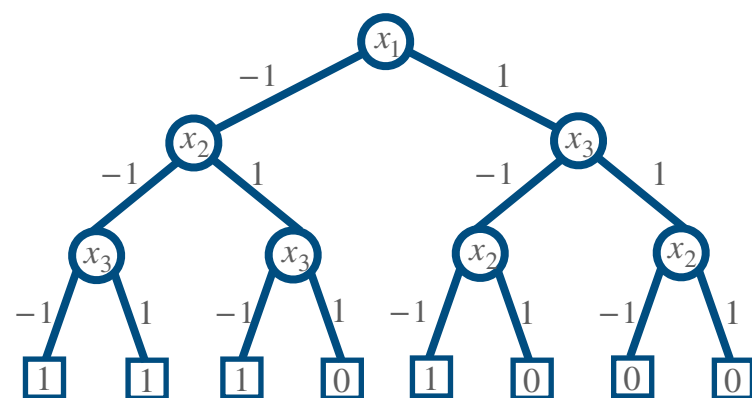
T computes $f : \{-1, 1\}^n \rightarrow \{0, 1, *\}$ with error ϵ if

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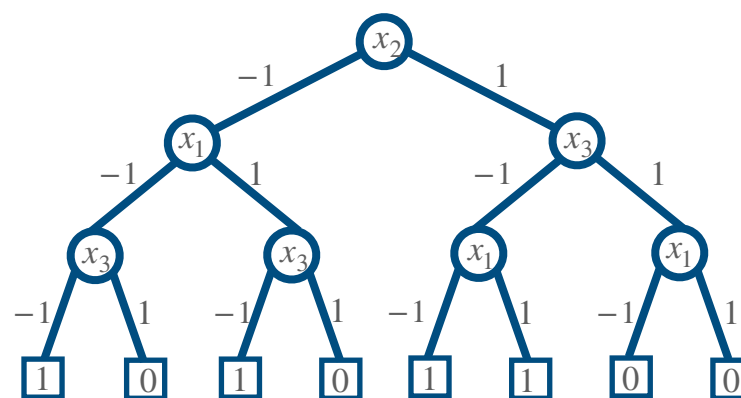
Query complexity



Randomized



T_1



T_2

...

$R_\epsilon(f)$ = minimum depth of a randomized decision tree for f with error ϵ .

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

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query
index

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

workspace

query index

The diagram shows the equation $|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$. The term $|i\rangle$ is circled in red, and the term $|w\rangle$ is circled in blue. The word "workspace" is written in blue above the blue circle, and the words "query index" are written in red below the red circle.

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

↓

$$|\phi'\rangle = \sum_{i,w} a_{i,w} x_i |i\rangle |w\rangle$$

workspace
query index

Quantum query complexity

Quantum query

$$|\phi\rangle = \sum_{i,w} a_{i,w} \underbrace{|i\rangle}_{\text{query index}} \underbrace{|w\rangle}_{\text{workspace}}$$

$$\downarrow$$
$$|\phi'\rangle = \sum_{i,w} a_{i,w} x_i |i\rangle |w\rangle$$

can access all x_i in a single query!

Quantum speedups

Query model captures nearly all quantum breakthroughs:

Deutsch-Jozsa's algorithm

Bernstein-Vazirani's algorithm

Simon's algorithm

Shor's factoring algorithm

Grover's search

.....

Quantum speedups

Reference	Randomized	Quantum
Simon 97	$\Omega(\sqrt{n})$	$O(\log n)$

Largest possible separation?

[Buhrman et al. 02, Aaronson-Ambainis 15]

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$$R(f) = \Omega(n), Q(f) = O(1)$$

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$$\textcolor{red}{\cancel{R(f) = \Omega(n), Q(f) = O(1)}}$$

Impossible!

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Optimal

Our results

Theorem.

Let k be any positive integer, $k \leq \frac{1}{3} \log n$. Then there is $f_k : \{-1, 1\}^n \rightarrow \{0, 1, *\}$ such that

$$Q_{\frac{1}{2} - \frac{1}{2^{k+4}}}(f_k) \leq \left\lceil \frac{k}{2} \right\rceil,$$

$$R_{\frac{1}{2^{k+1}}}(f_k) \geq \Omega \left(\frac{n^{1 - \frac{1}{k}}}{(\log n)^{2 - \frac{1}{k}}} \right).$$

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$$Q_{1/3}(f_k) = O(k4^k),$$

$$R_{1/3}(f_k) = \Omega \left(\frac{n^{1-\frac{1}{k}}}{k(\log n)^{2-\frac{1}{k}}} \right).$$

Our results

Corollary I.

For any $\epsilon > 0$, there is $f: \{-1, 1\}^n \rightarrow \{0, 1, *\}$ with

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Take $k = 1 + \lceil 1/\epsilon \rceil$

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Corollary 2.

For any monotone $\alpha: \mathbb{N} \rightarrow \mathbb{N}$, there is $f: \{-1, 1\}^n \rightarrow \{0, 1, *\}$ with

$$Q_{1/3}(f) \leq \alpha(n),$$

$$R_{1/3}(f) = n^{1-o(1)}.$$

Our results

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Take $k = k(n)$ an arbitrarily slow-growing function, e.g.
 $k = \log \log \log n$.

Our results: total functions

Reference	Randomized vs. Quantum
Grover 69, BBBV 97	$R(f) = \Omega(Q(f)^2)$
Beals et al. 01	$R(f) = O(Q(f)^6)$

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Partial functions $f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,*\}$,

Reference	Classical	Quantum
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Raz 99	$R(f) = \tilde{\Omega}(n^{1/4})$	$O(\log n)$
Klartag-Regev 10	$R(f) = \tilde{\Omega}(n^{1/3})$	$O(\log n)$
Aaronson-Ambainis 15	$R(f) = \tilde{\Omega}(n^{1/2})$	$O(\log n)$
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near-optimal

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near-optimal

} lifting from query model

Our results: communication

Total functions $f: \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$,

Reference	Classical vs. Quantum
Buhrman et al. 98, Razborov 02	$R(f) \geq \Omega(Q(f)^2)$
Aaronson et al. 15	$R(f) \geq \tilde{\Omega}(Q(f)^{5/2})$
Tal 19	$R(f) \geq \Omega(Q(f)^{8/3-o(1)})$
Our work	$R(f) \geq \Omega(Q(f)^{3-o(1)})$

Our results: Fourier weight

Theorem

For any decision tree $g : \{-1, 1\}^n \rightarrow \{0, 1\}$ of depth d ,

$$\sum_{\substack{S \subseteq \{1, 2, \dots, n\}: \\ |S| = \ell}} |\hat{g}(S)| \leq c^\ell \sqrt{\binom{d}{\ell} (1 + \log n)^{\ell-1}}.$$

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- Essentially optimal
- Settles conjecture by Tal (2019)
- Previous bounds trivial already at $\ell \geq \sqrt{d}$

Independent work by Bansal & Sinha

Bansal–Sinha

Our work

Independent work by Bansal & Sinha

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stochastic calculus

Our work

Fourier analysis

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stochastic calculus

- advanced machinery

Our work

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Our work

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- elementary
- optimal Fourier weight of decision trees

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explicit

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existential

The problem: correlation

Rorrelation

$U \in \mathbb{R}^{n \times n}$, orthogonal matrix

$x_1, x_2, \dots, x_k \in \{-1, 1\}^n$

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$$\phi_{n,k,U}(x_1, x_2, \dots, x_k) = \frac{1}{n} \mathbf{1}^T D_{x_1} U D_{x_2} U \dots U D_{x_k} \mathbf{1}$$

Rorrelation

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$$f_{n,k,U}(x_1, x_2, \dots, x_k) = \begin{cases} 1 & \phi_{n,k,U} > 2^{-k}, \\ 0 & |\phi_{n,k,U}| \leq 2^{-k-1}, \\ * & \text{otherwise.} \end{cases}$$

Rorrelation: quantum algorithms

$$\phi_{n,k,U}(x_1, x_2, \dots, x_k) = \frac{1}{n} \mathbf{1}^T D_{x_1} U D_{x_2} U \dots U D_{x_k} \mathbf{1}$$

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Theorem (Aaronson-Ambainis, Tal).

There is a quantum algorithm using $\lceil k/2 \rceil$ queries that accepts x with probability

$$\frac{\phi_{n,k,U}(x) + 1}{2}.$$

Rorrelation: classical lower bound
—the “indistinguishability” argument

Correlation: classical lower bound

—the “indistinguishability” argument

$\mathcal{U}_{n,k}$ = uniform distribution

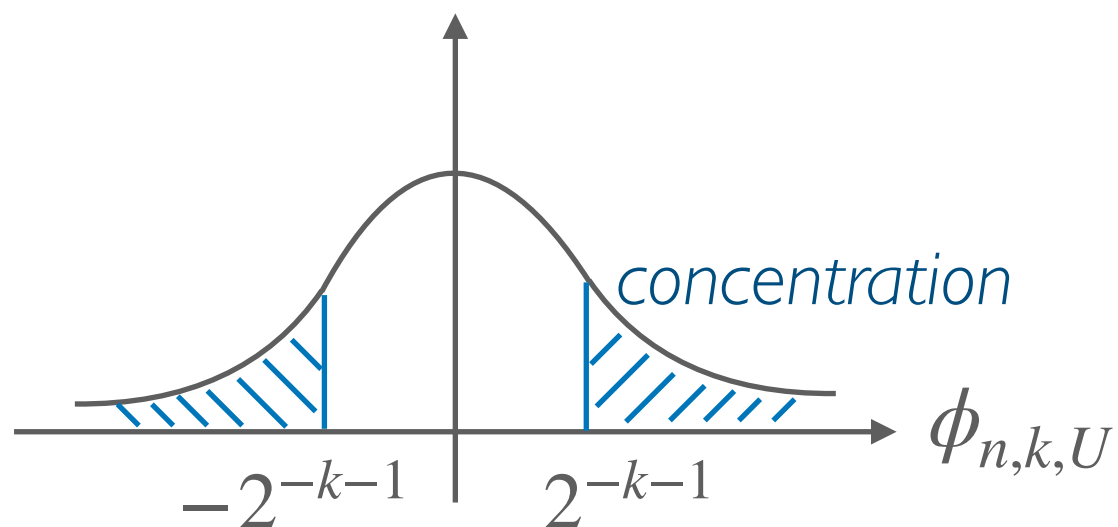
$\mathcal{D}_{n,k,U}$ = correlated distribution

Correlation: classical lower bound

—the “indistinguishability” argument

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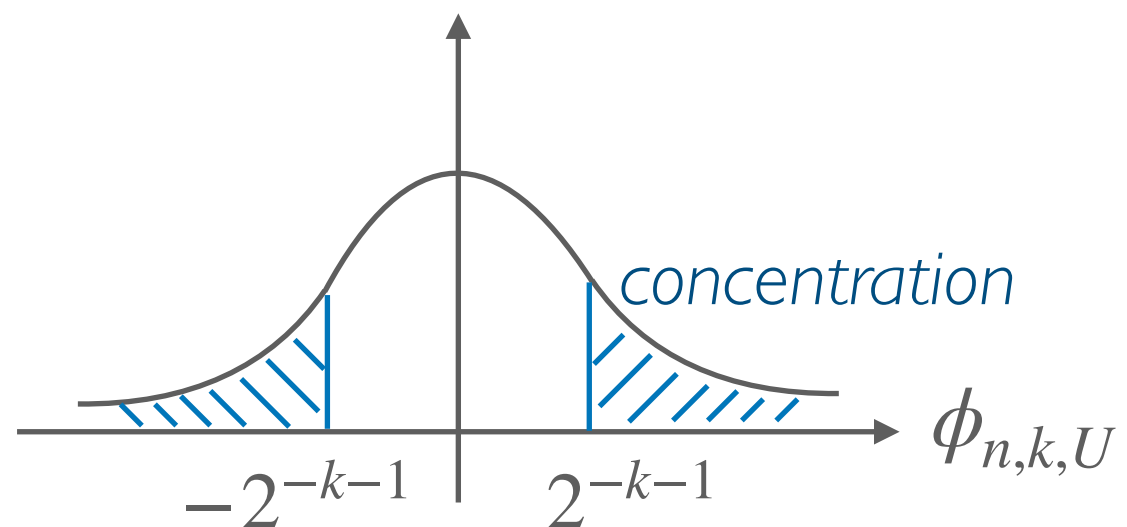


$$\mathbf{P}_{\mathcal{U}_{n,k}}[\phi > 2^{-k-1}] < 2^{-k-1}$$

Correlation: classical lower bound

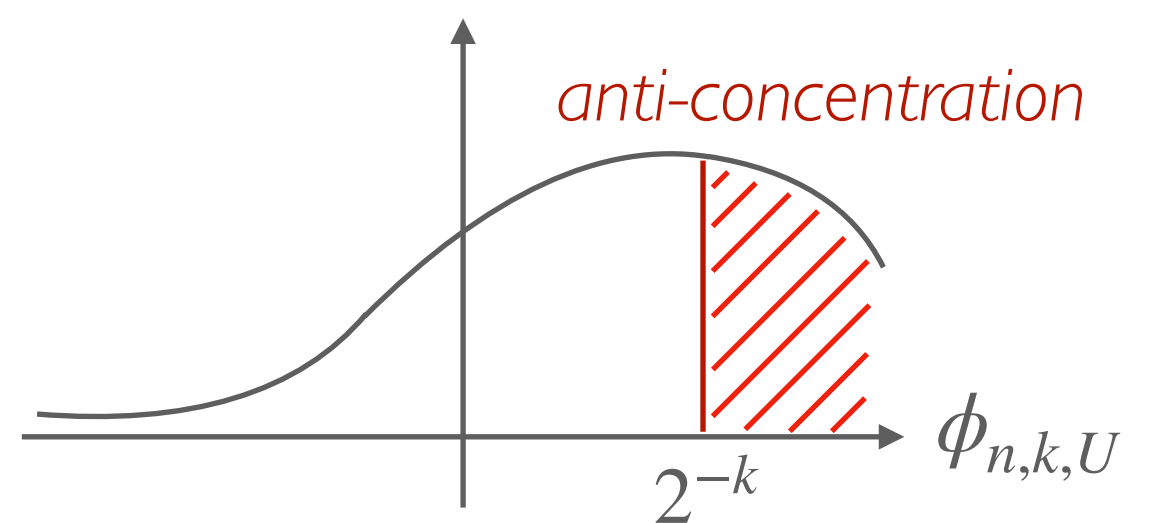
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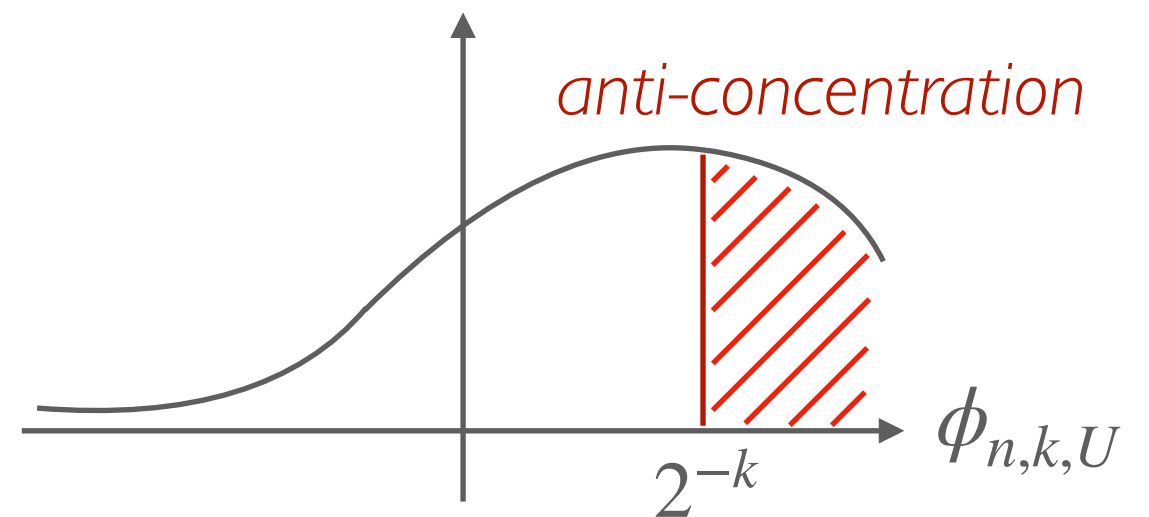
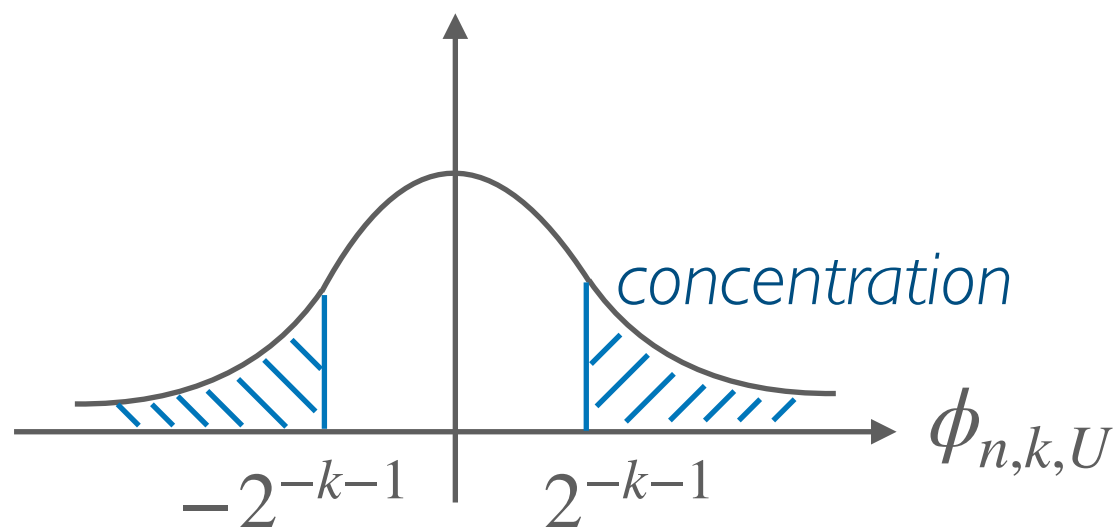
$$\mathbf{P}_{\mathcal{D}_{n,k,U}}[\phi \geq 2^{-k}] \geq 2^{-k}$$

Correlation: classical lower bound

—the “indistinguishability” argument

$\mathcal{U}_{n,k}$ = uniform distribution

$\mathcal{D}_{n,k,U}$ = correlated distribution



Thus, for any randomized query algorithm g of error ϵ ,

$$\mathbf{E}_{\mathcal{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathcal{U}_{n,k}} g(x) \geq 2^{-k-1} - 2\epsilon .$$

Correlation: classical lower bound

—the “indistinguishability” argument

$$\mathbf{E}_{\mathcal{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathcal{U}_{n,k}} g(x)$$

Correlation: classical lower bound

—the “indistinguishability” argument

$$\begin{aligned} & \mathbf{E}_{\mathcal{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathcal{U}_{n,k}} g(x) \\ & \leq \left| \mathbf{E}_{\mathcal{D}_{n,k,U}} \sum_S \hat{g}(S) \chi_S - \mathbf{E}_{\mathcal{U}_{n,k}} \sum_S \hat{g}(S) \chi_S \right| \end{aligned}$$

Correlation: classical lower bound

—the “indistinguishability” argument

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Correlation: classical lower bound

—the “indistinguishability” argument

$$\begin{aligned}
 & \mathbf{E}_{\mathcal{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathcal{U}_{n,k}} g(x) \\
 & \leq \left| \mathbf{E}_{\mathcal{D}_{n,k,U}} \sum_S \hat{g}(S) \chi_S - \mathbf{E}_{\mathcal{U}_{n,k}} \sum_S \hat{g}(S) \chi_S \right| \\
 & = \left| \sum_{S \neq \emptyset} \hat{g}(S) \mathbf{E}_{\mathcal{D}_{n,k,U}} \chi_S \right| \\
 & \leq \sum_{\ell=1}^n \sum_{|S|=\ell} |\hat{g}(S)| \cdot \max_{T=\ell} |\mathbf{E}_{\mathcal{D}_{n,k,U}} \chi_S|
 \end{aligned}$$

Correlation: classical lower bound

—the “indistinguishability” argument

$$\mathbf{E}_{\mathcal{D}_{n,k,U}} g(x) - \mathbf{E}_{\mathcal{U}_{n,k}} g(x)$$

$$\leq \left| \mathbf{E}_{\mathcal{D}_{n,k,U}} \sum_S \hat{g}(S) \chi_S - \mathbf{E}_{\mathcal{U}_{n,k}} \sum_S \hat{g}(S) \chi_S \right|$$

$$= \left| \sum_{S \neq \emptyset} \hat{g}(S) \mathbf{E}_{\mathcal{D}_{n,k,U}} \chi_S \right|$$

$$\leq \sum_{\ell=1}^n \sum_{|S|=\ell} |\hat{g}(S)| \cdot \max_{T=\ell} |\mathbf{E}_{\mathcal{D}_{n,k,U}} \chi_S|$$

$\ell = k$

$$\stackrel{[\text{Tal}]}{\leq} O \left(\frac{\ell \log n}{n} \right)^{\frac{\ell}{2} \frac{k-1}{k}}$$

Rorrelation: classical lower bound

—the “indistinguishability” argument

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$$= \left| \sum_{S \neq \emptyset} \hat{g}(S) \mathbf{E}_{\mathcal{D}_{n,k,U}} \chi_S \right|$$

$$\leq \sum_{\substack{\ell=1 \\ \ell=k}}^n \sum_{|S|=\ell} |\hat{g}(S)| \cdot \max_{T=\ell} |\mathbf{E}_{\mathcal{D}_{n,k,U}} \chi_S|$$

$$\stackrel{[\text{Tal}]}{\leq} O \left(\frac{\ell \log n}{n} \right)^{\frac{\ell}{2} \frac{k-1}{k}}$$

We prove: $\leq c^\ell \sqrt{\binom{d}{\ell}} (\ln en)^{\ell-1}$

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We prove: $\leq c^\ell \sqrt{\binom{d}{\ell}} (\ln en)^{\ell-1}$

Therefore,

$$R_{2-o(k)}(f_k) = \tilde{\Omega}(n^{1-\frac{1}{k}}). \blacksquare$$

Fourier weight of decision trees

Fourier weight of decision trees

Main Theorem.

For any decision tree $T : \{-1, 1\}^n \rightarrow \{0, 1\}$ of depth d ,

$$\sum_{\substack{S \subseteq \{1, 2, \dots, n\}: \\ |S| = \ell}} |\hat{T}(S)| \leq c^\ell \sqrt{\binom{d}{\ell} (1 + \log n)^{\ell-1}}.$$

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Fix any decision tree $T : \{-1, 1\}^n \rightarrow \{-1, 0, 1\}$ of depth d , and $\mathbf{P}[T(x) \neq 0] = p$. Then

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$$\Lambda_{m, \ell} = \begin{cases} 0, & \text{if } p = 0, \\ p \sqrt{\left(\frac{1}{\ell} \ln \frac{e^\ell m^{\ell-1}}{p}\right)^\ell}, & \text{if } 0 < p \leq 1/m, \\ p \sqrt{\left(\ln \frac{e}{p}\right) (\ln em)^{\ell-1}}, & \text{if } 1/m \leq p \leq 1. \end{cases}$$

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**increasing,
concave**

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**increasing,
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Known results

Nonadaptive case:

$$\|L_\ell T\| \leq \sqrt{\binom{d}{\ell} \sum_{S:|S|=\ell} \hat{T}(S)^2} \leq \sqrt{\binom{d}{\ell}}.$$

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Known results

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By negating variables, we can assume $\hat{T}(i) \geq 0$. Then

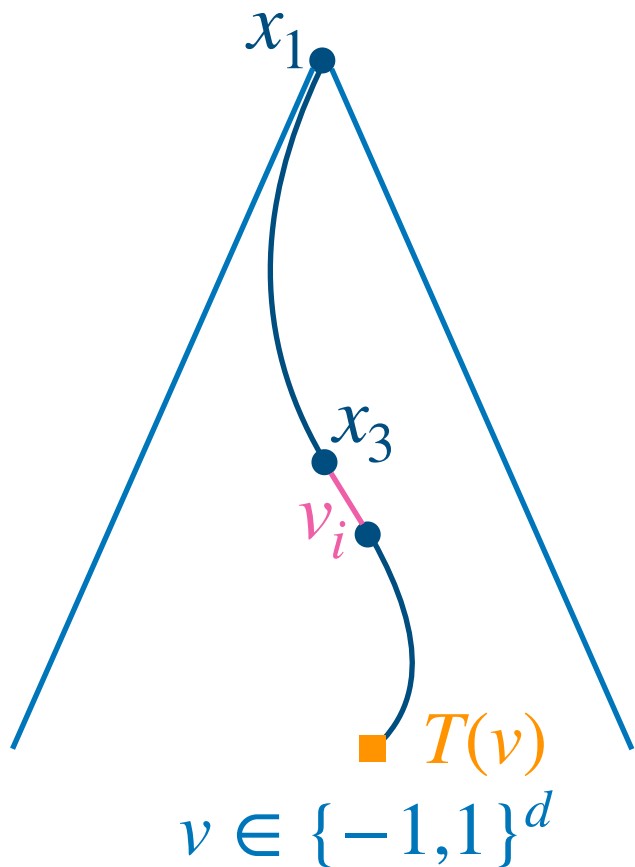
$$\|L_1 T\| = \sum_{i=1}^n \hat{T}(i) = \mathbf{E}_{v \in \{-1,1\}^d} \left[T(v) \sum_{i=1}^d v_i \right]$$

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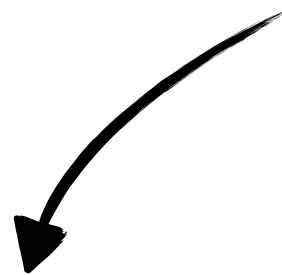


Known results

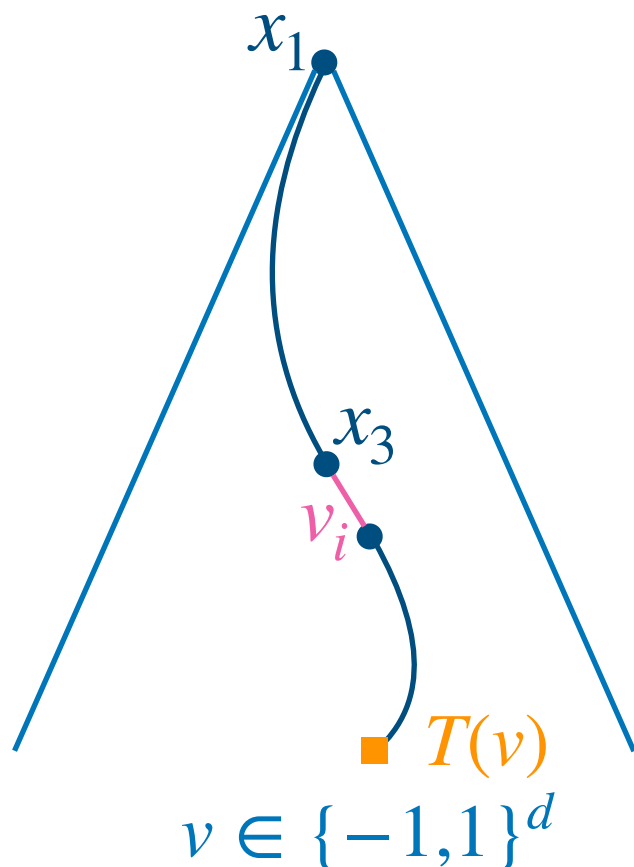
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Maximized when paths with larger sums have label 1

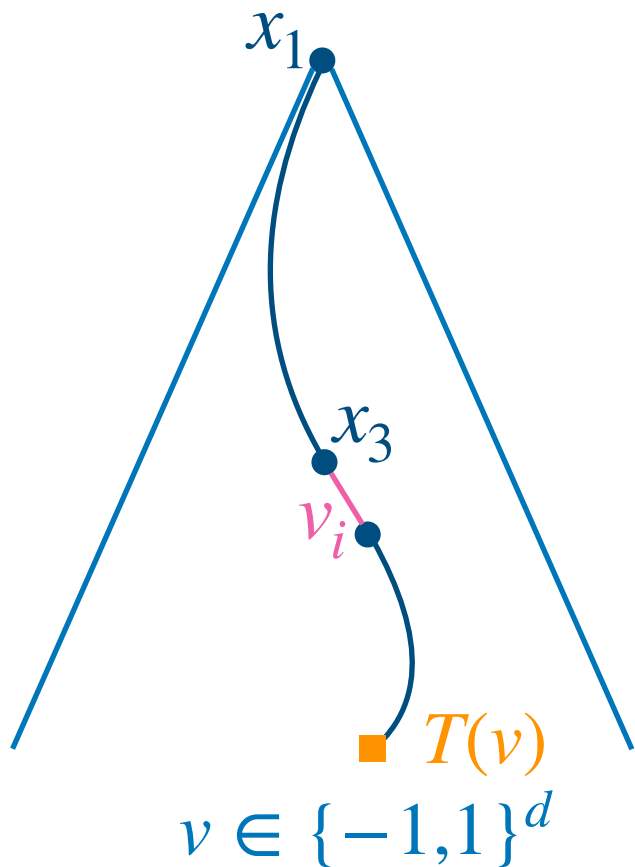


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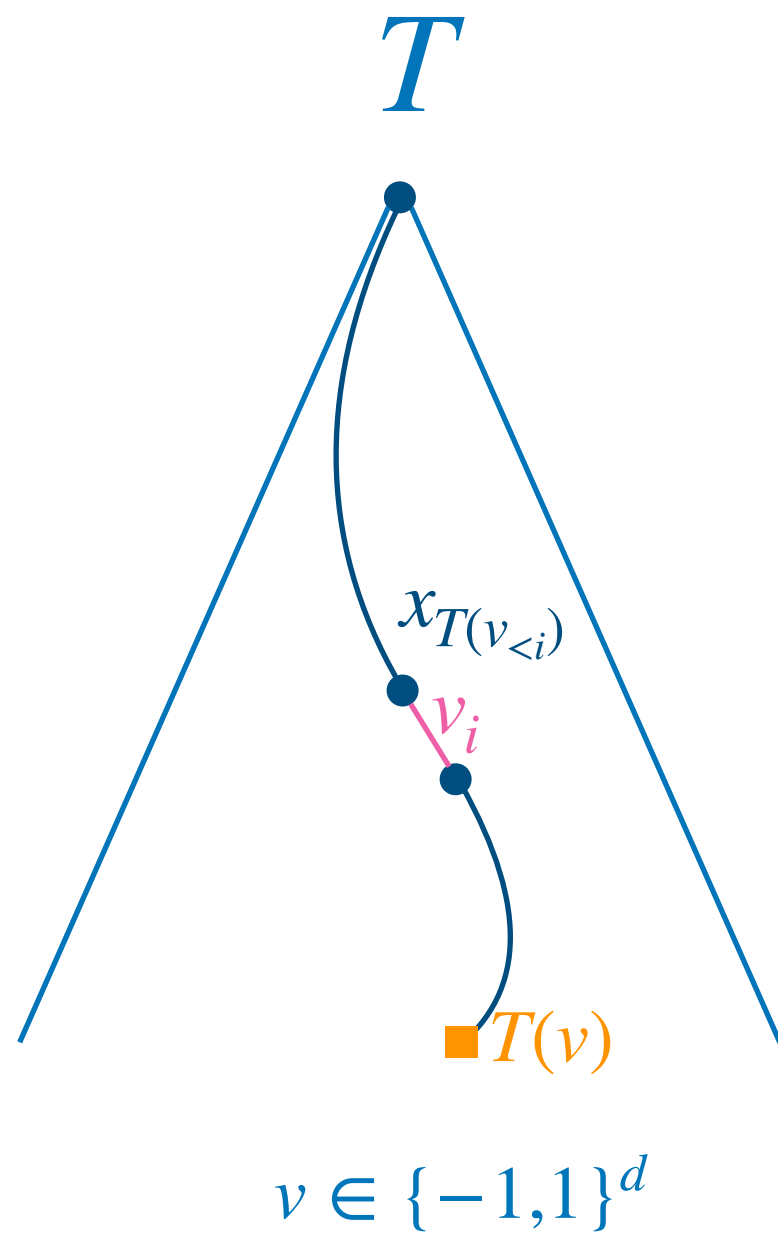


Theorem (O'Donnell-Servedio 07, Tal 19)

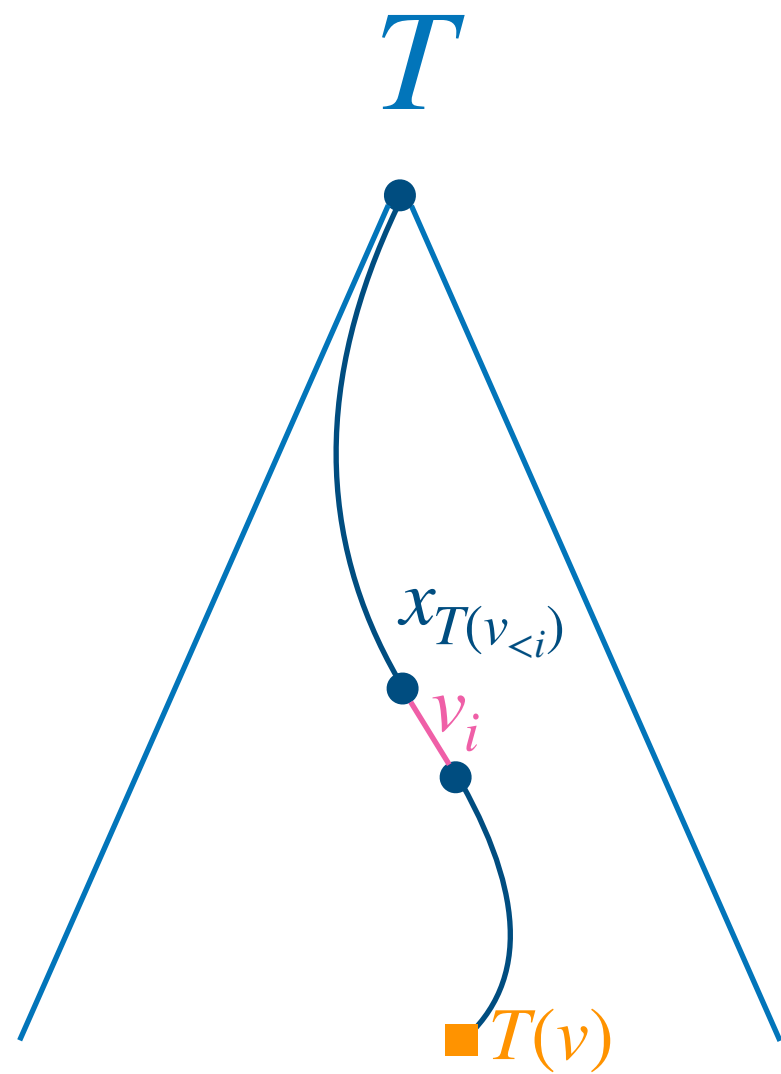
$$\|L_1 T\| \leq O \left(p \sqrt{d \ln \frac{e}{p}} \right).$$

Our approach

Our approach



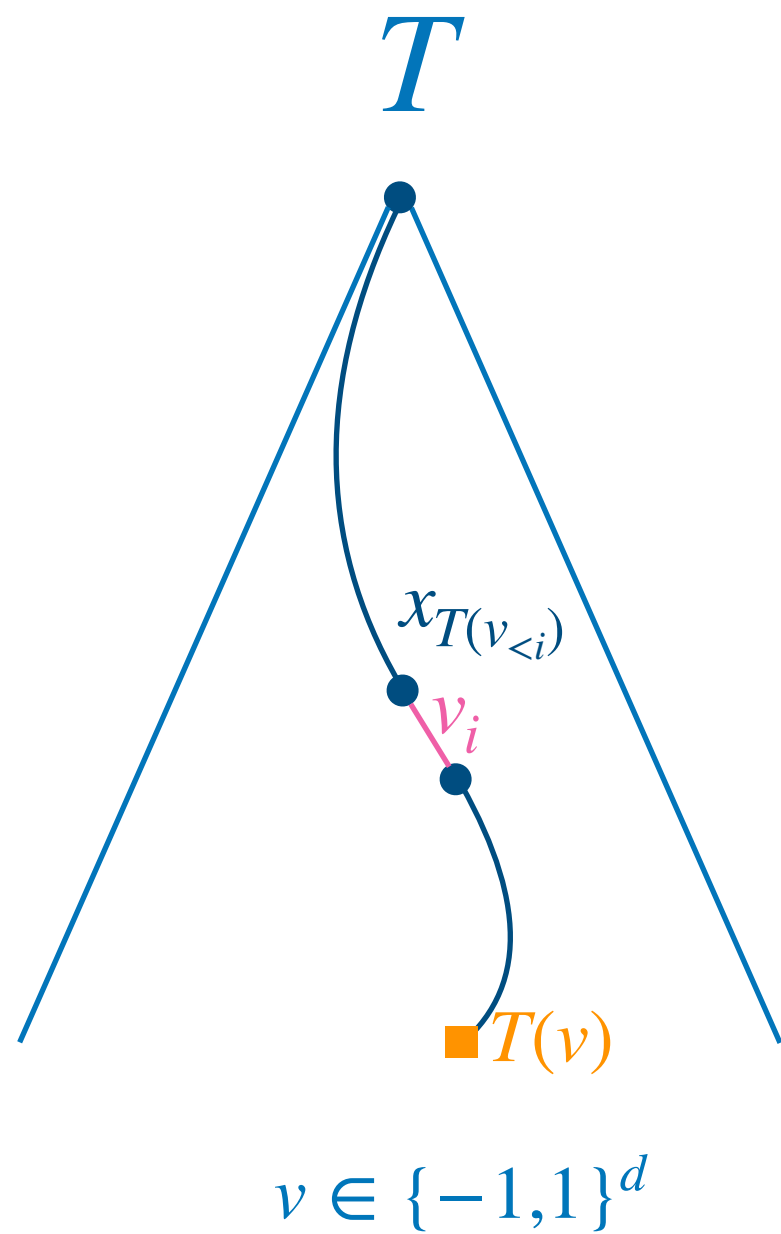
Our approach



$$v \in \{-1, 1\}^d$$

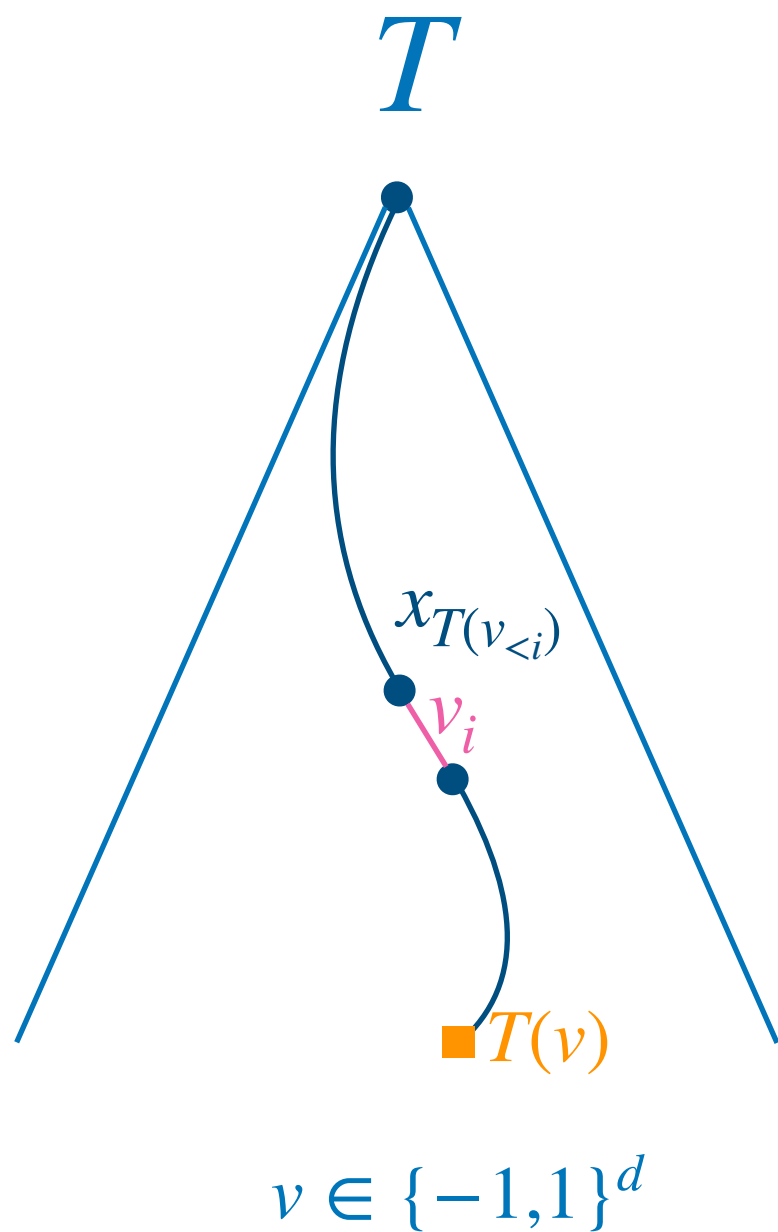
$$T(v) \prod_{i=1}^d \frac{1 + v_i x_{T(v_{<i})}}{2}$$

Our approach



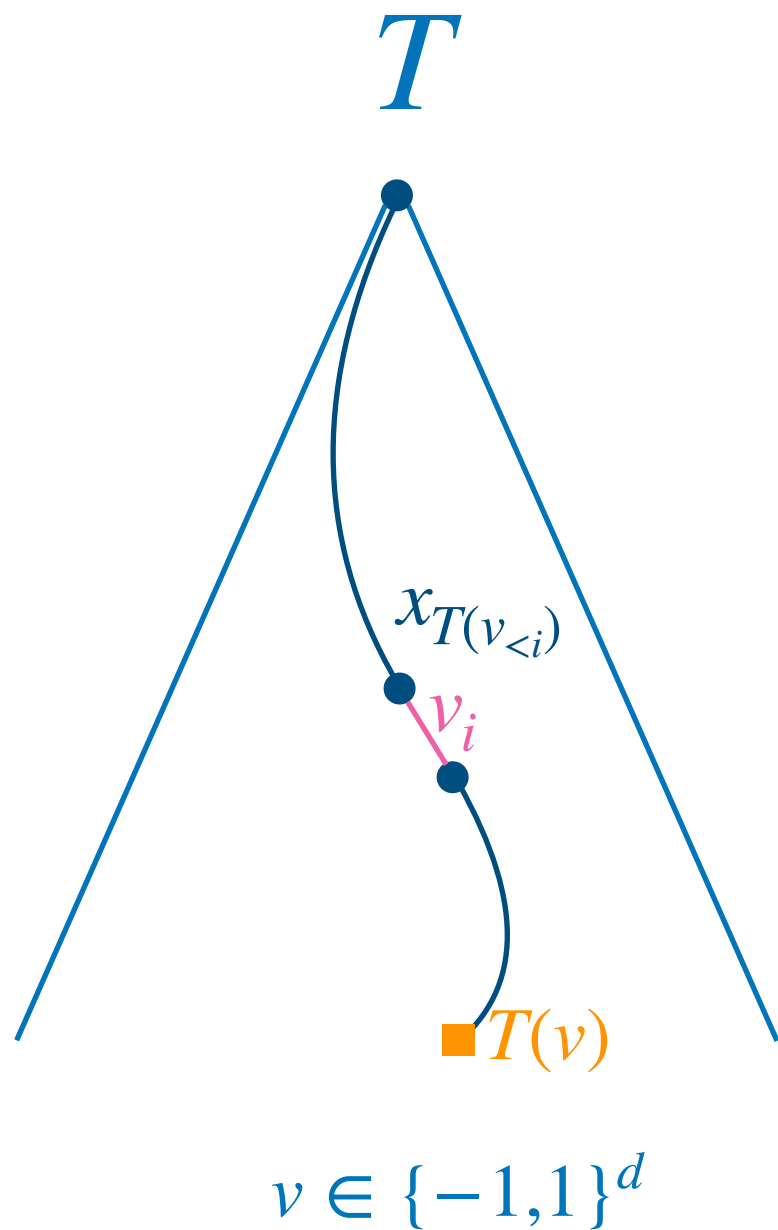
$$T = \sum_{v \in \{-1, 1\}^d} T(v) \prod_{i=1}^d \frac{1 + v_i x_{T(v_{<i})}}{2}$$

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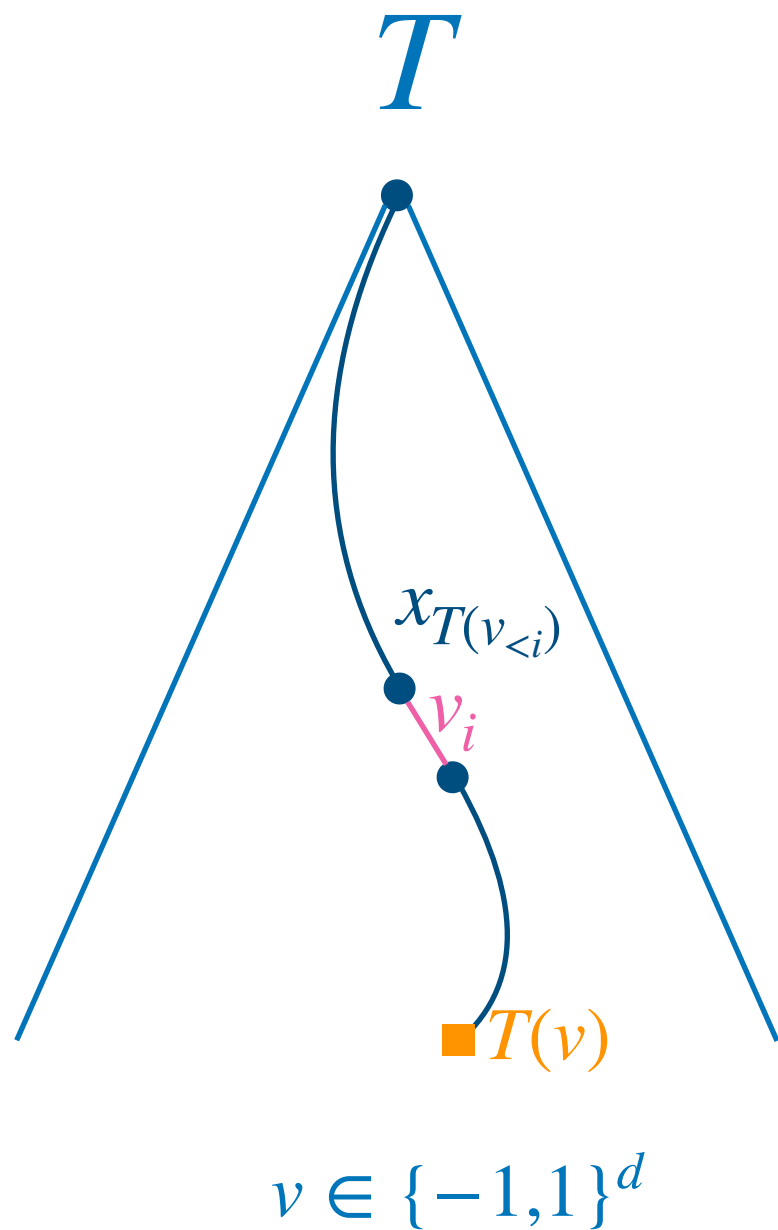
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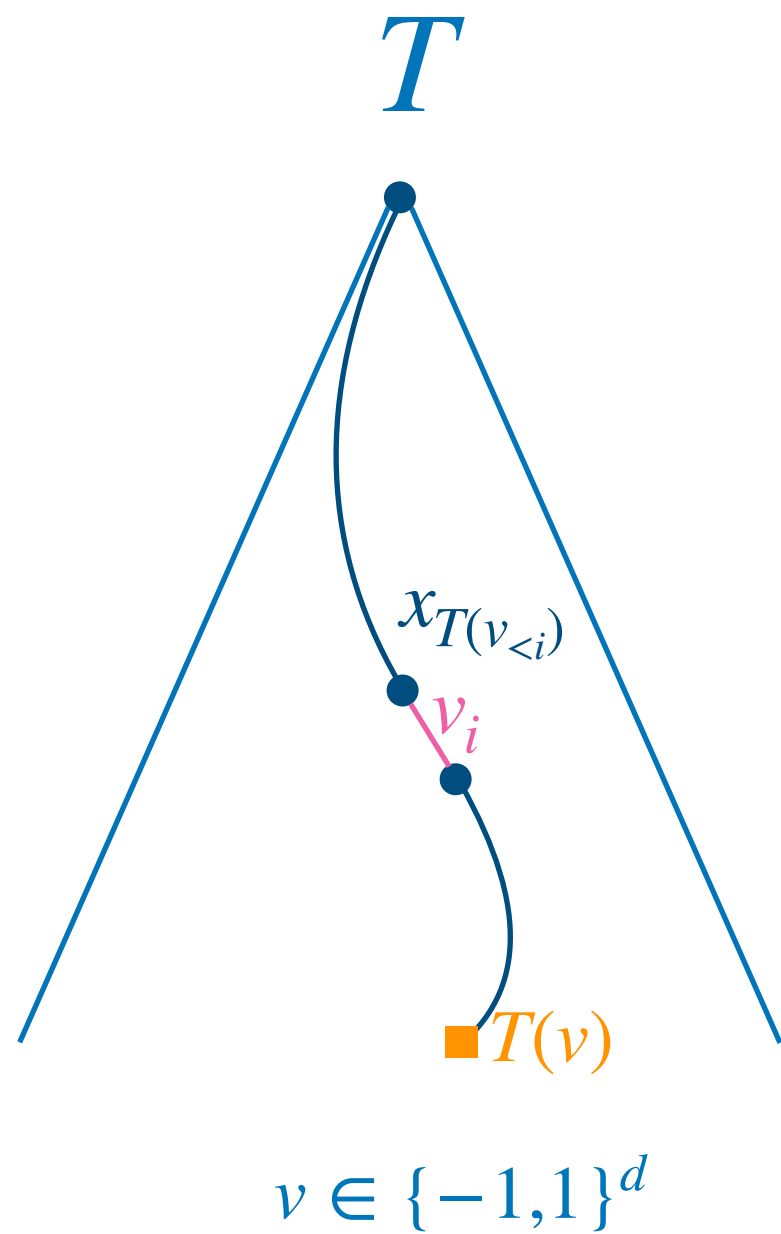
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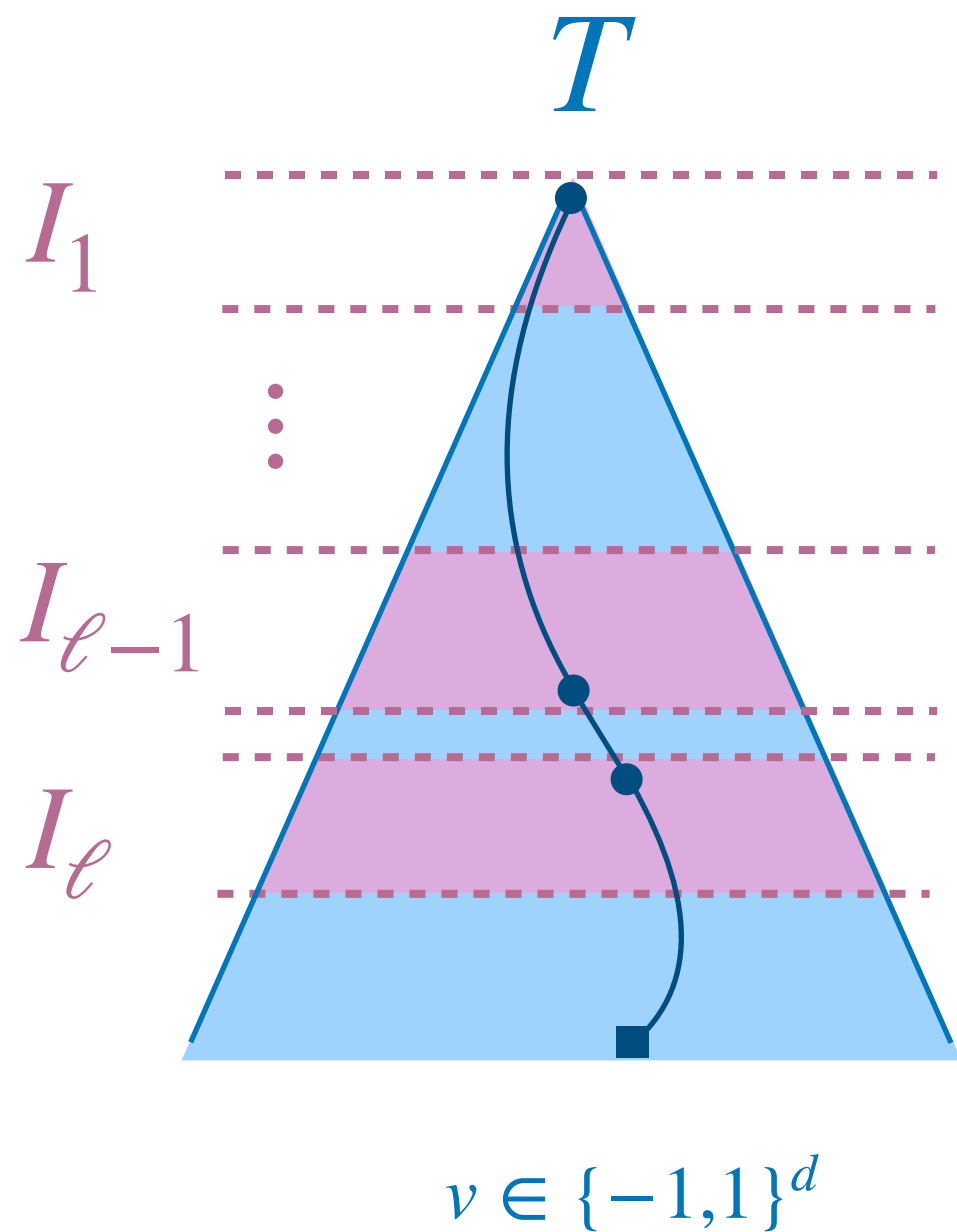
$$L_\ell T = \sum_{S \in \mathcal{P}_{d, \ell}} \sum_{v \in \{-1, 1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} \cdot$$

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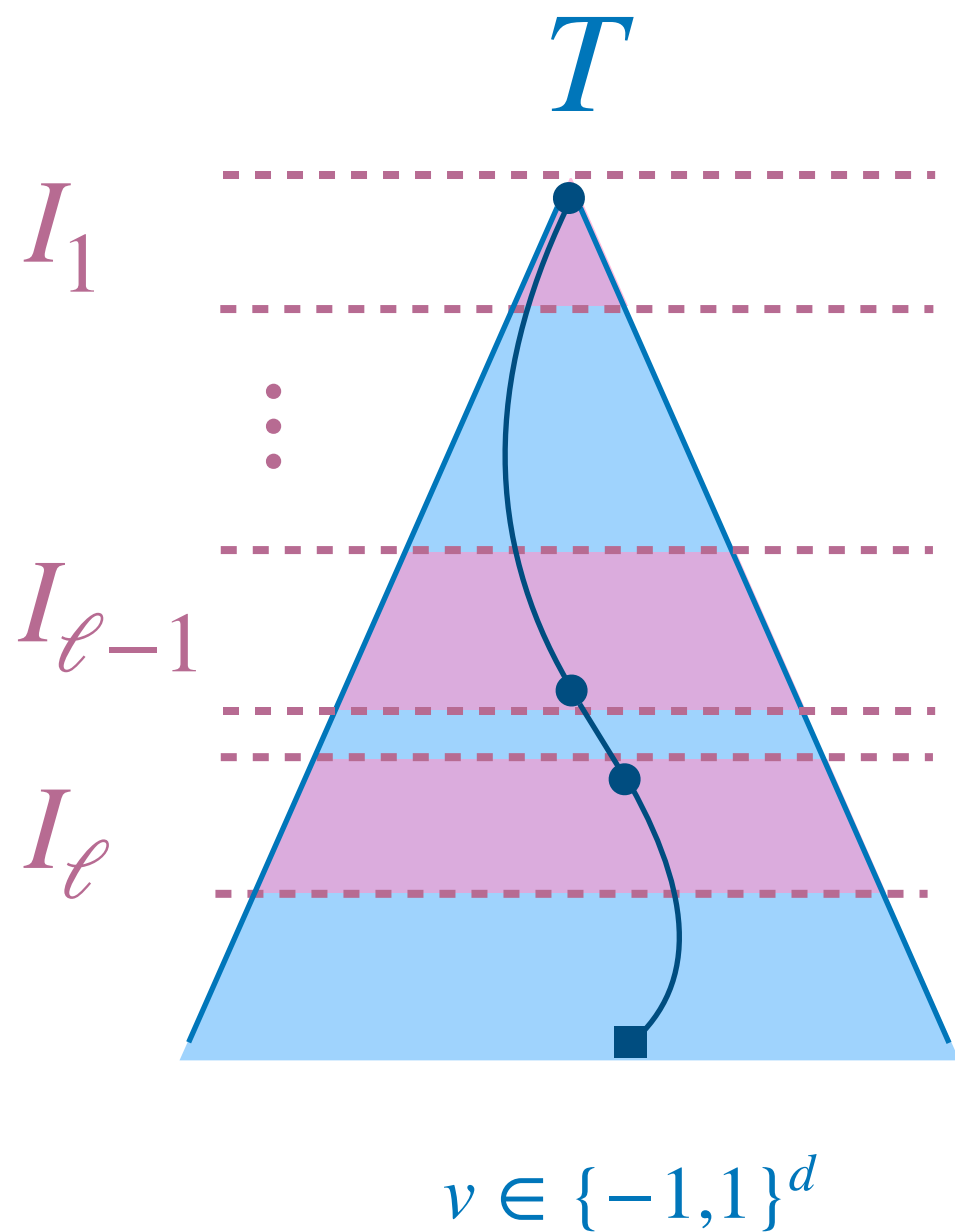
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Our approach



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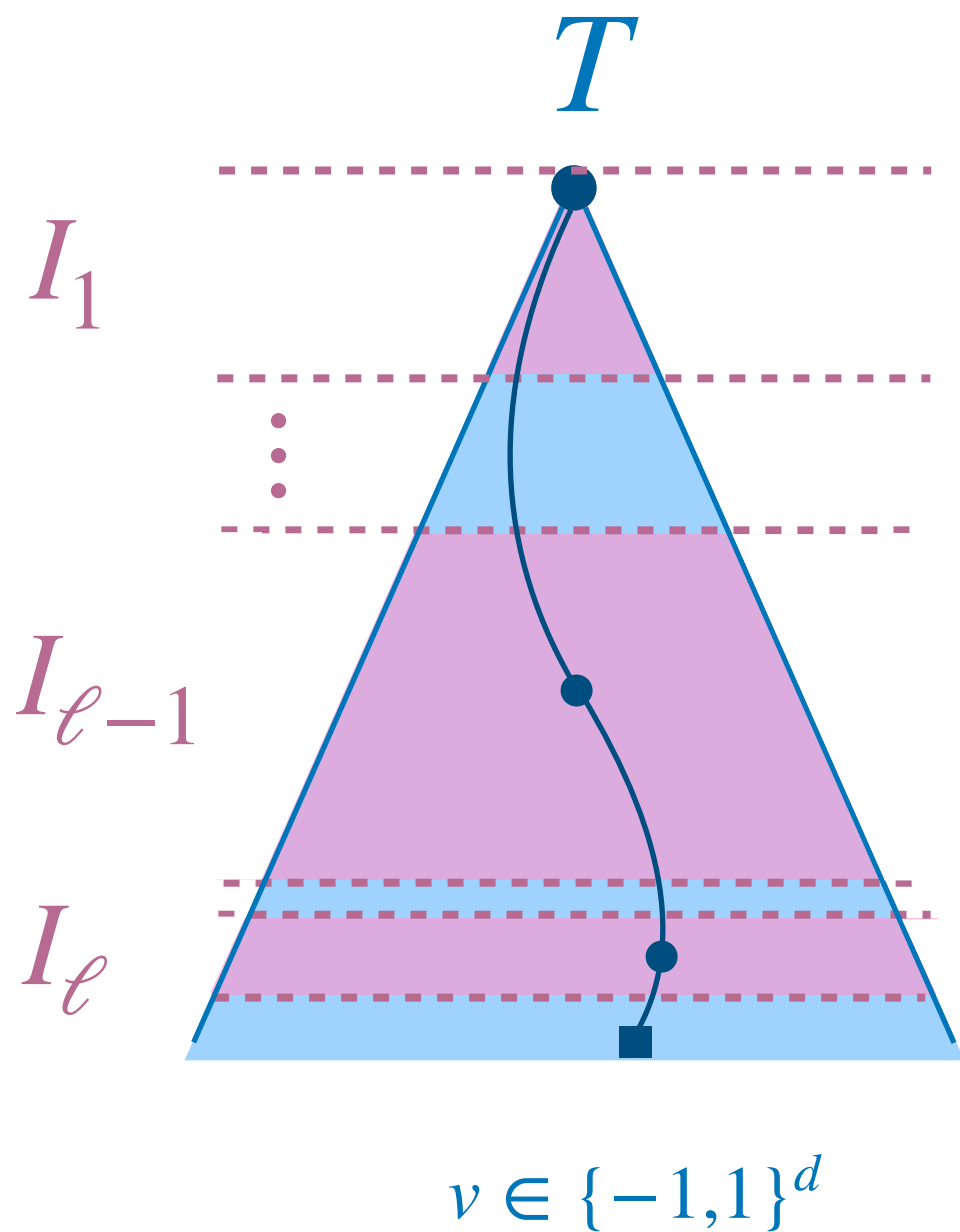
Our approach



$$L_{\ell} T = \sum_{S \in \mathcal{P}_{d, \ell}} \sum_{v \in \{-1, 1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{< i})}.$$

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Our approach

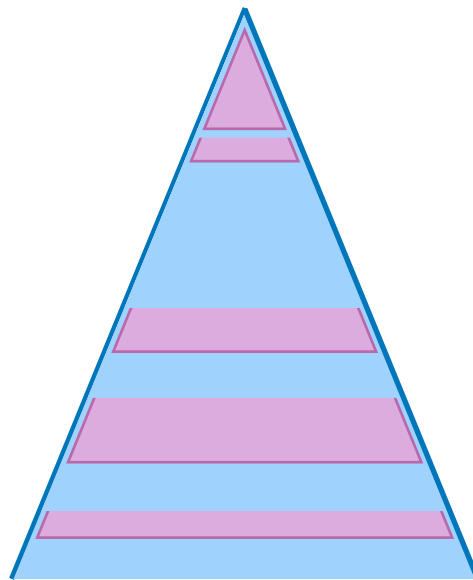


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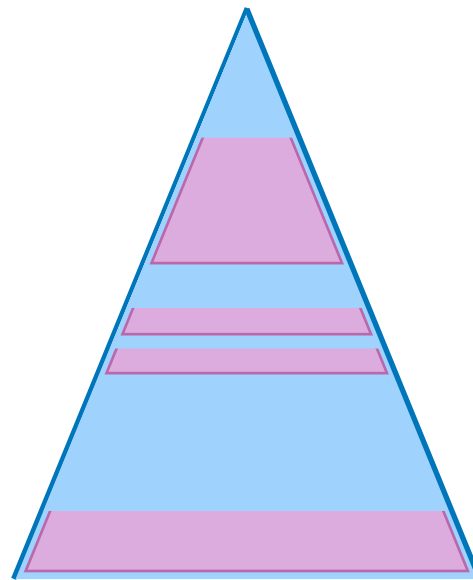
Our key idea

$$L_\ell T =$$



$$T|_{\mathcal{E}_1}$$

+

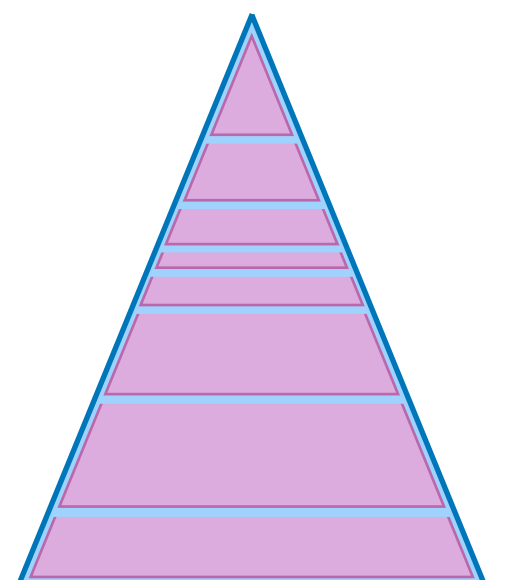


$$T|_{\mathcal{E}_2}$$

+

...

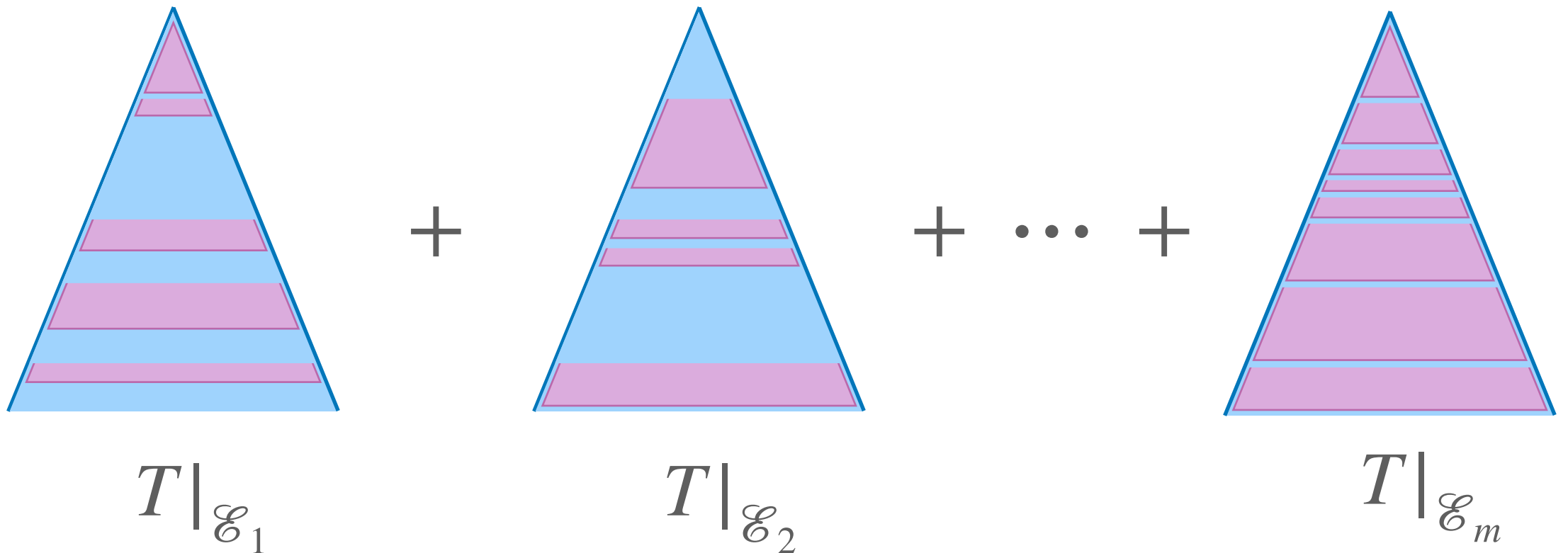
+



$$T|_{\mathcal{E}_m}$$

Our key idea

$$L_\ell T =$$



$$\|L_\ell T\| \leq \sum \|T|_{\mathcal{E}_i}\|. \text{ (Triangle-inequality)}$$

Our proof

$$\|L_\ell T\| \leq \sum_i \|T|_{\mathcal{E}_i}\|.$$

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Theorem I.

For some absolute constant c , and any elementary family $\mathcal{E} = I_1 * I_2 * \dots * I_\ell$,

$$\|T|_{\mathcal{E}}\| \leq c^\ell \sqrt{|\mathcal{E}|} \Lambda_{n^2, \ell}(\text{dns}(T)).$$

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Theorem 2.

$\mathcal{P}_{d, \ell}$ can be partitioned into elementary families $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m$ s.t. for some const C ,

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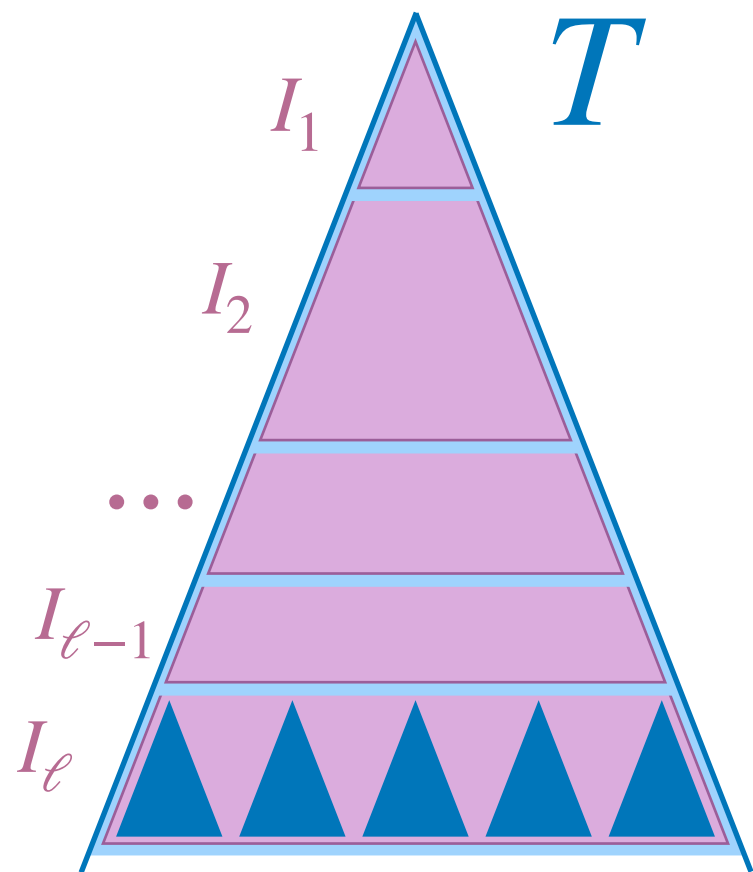
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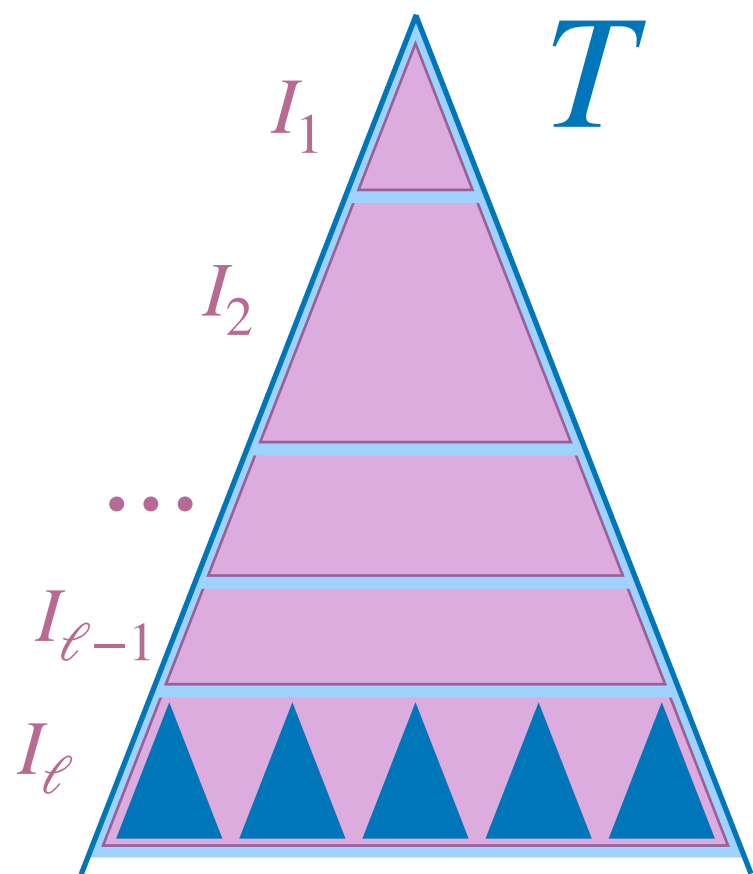
Fourier weight of decision trees

*Analysis of the individual parts
(Theorem 1.)*

Analysis of the individual part



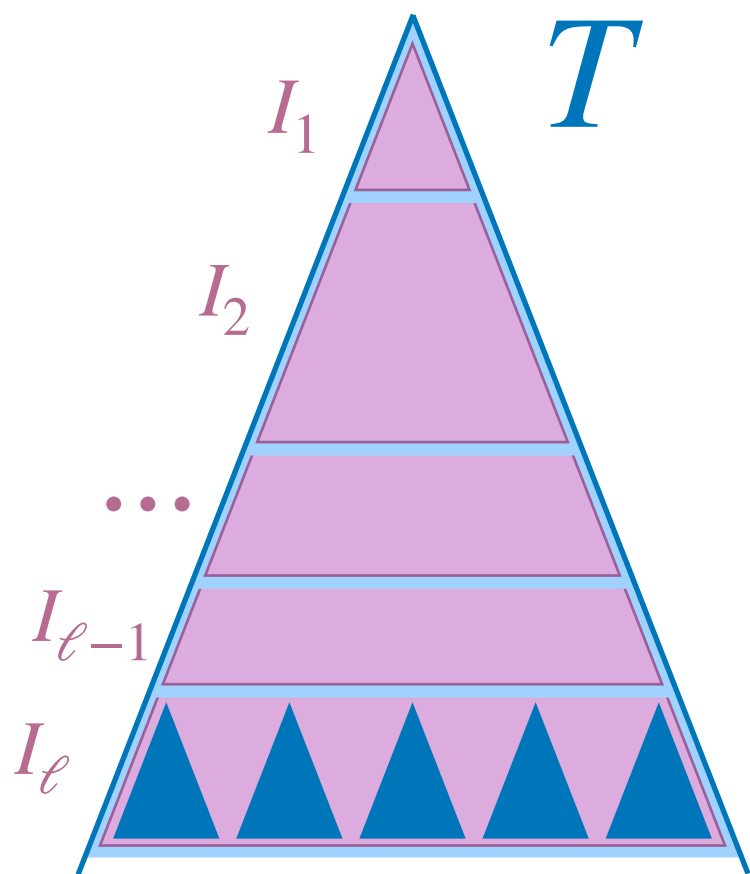
Analysis of the individual part



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Analysis of the individual part



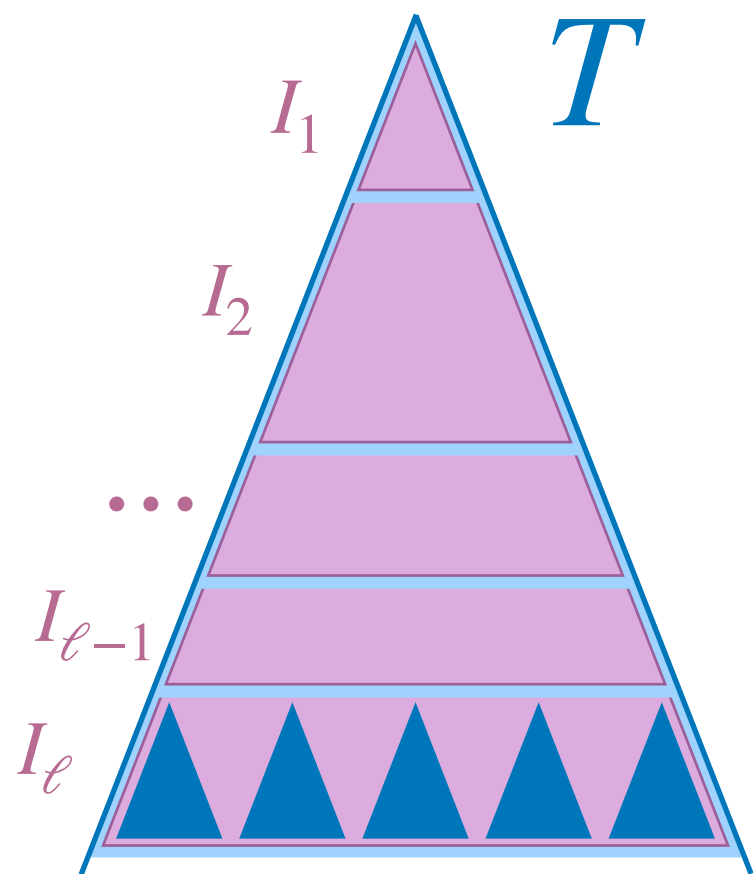
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Preview: Induction on ℓ .

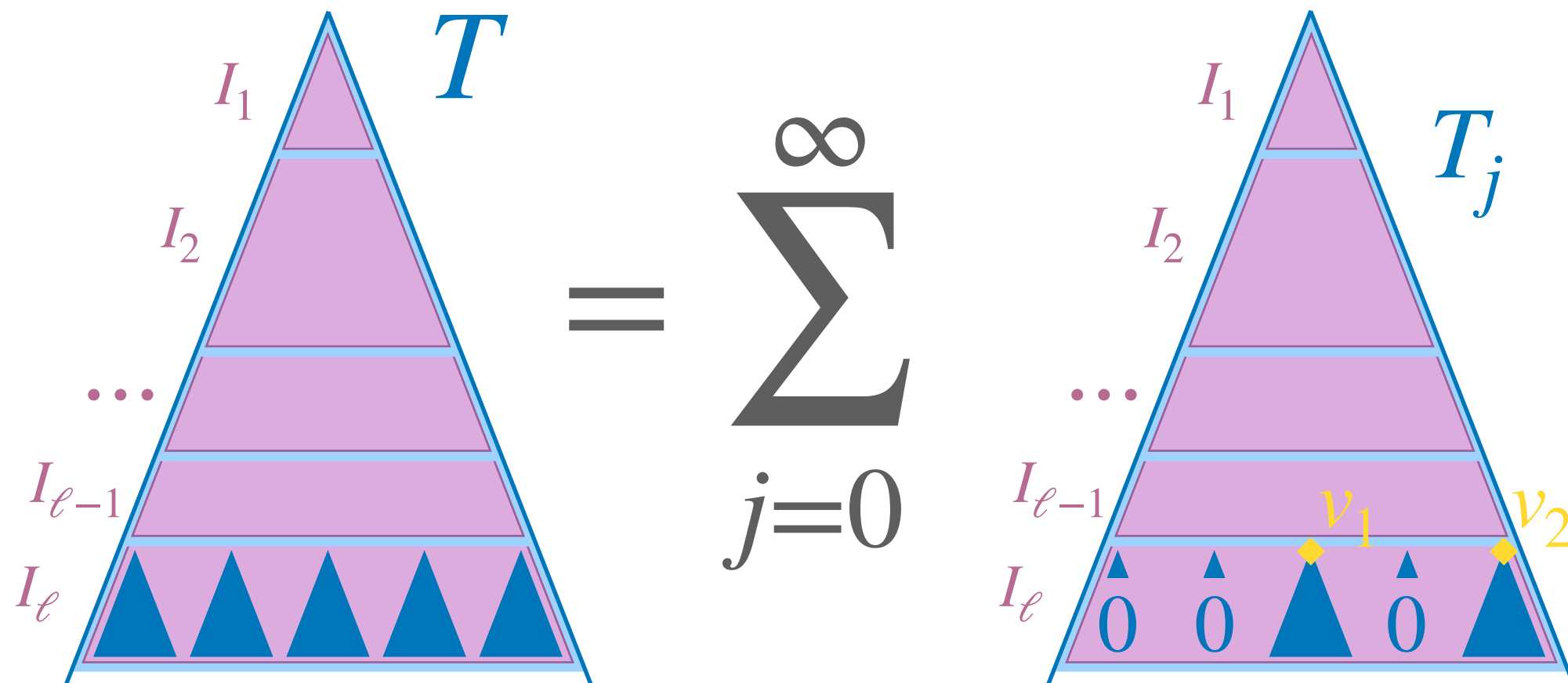
Analysis of the individual part

I. “Bucketing”



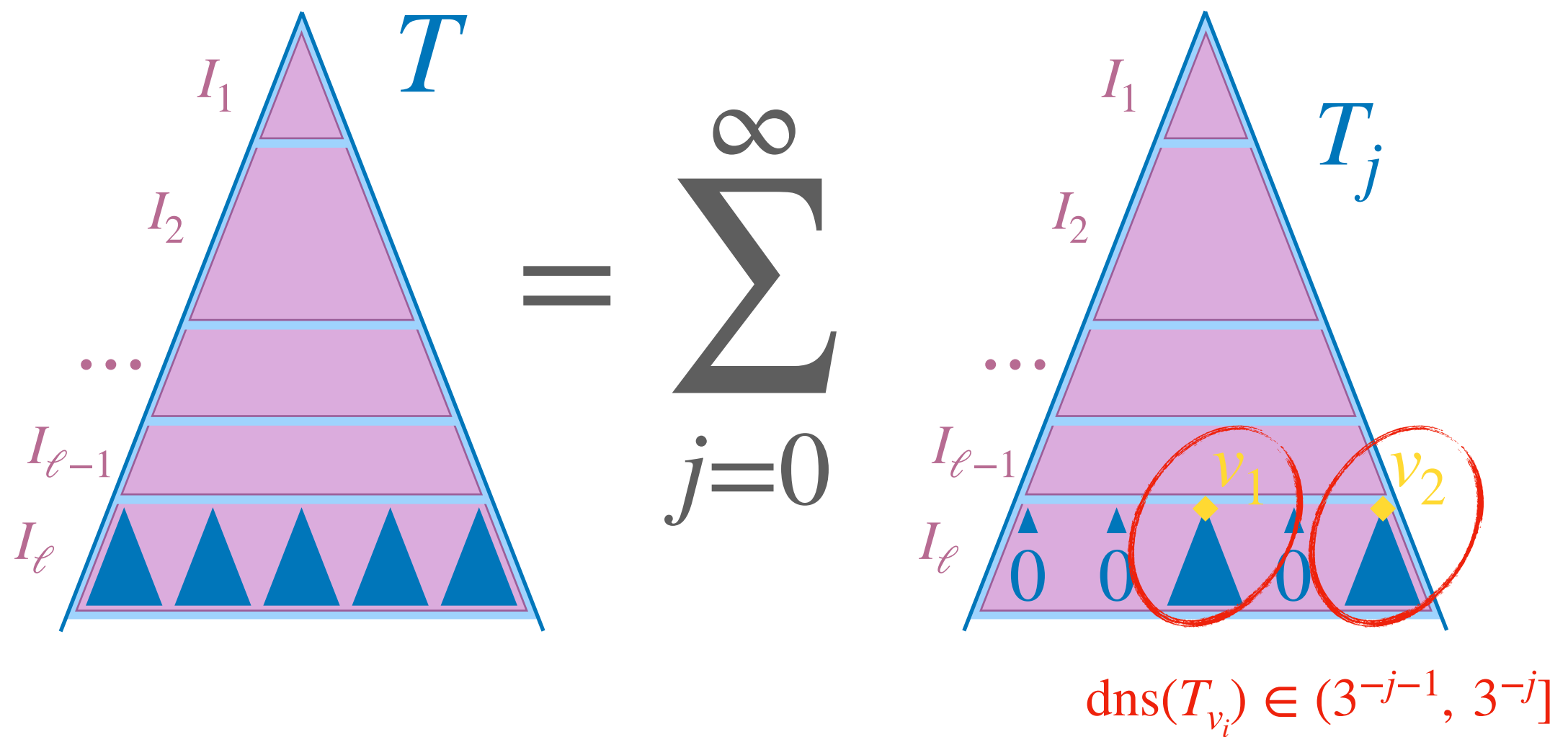
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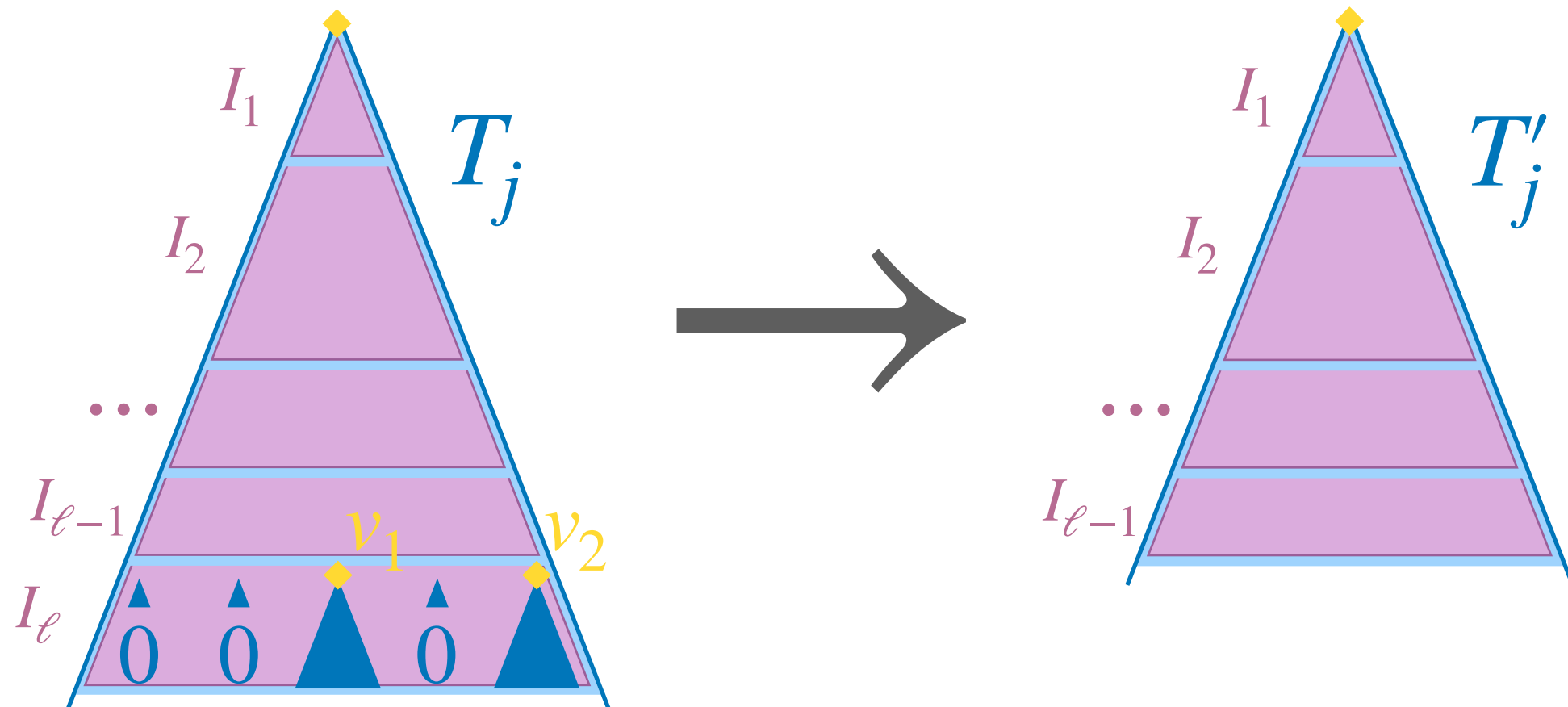
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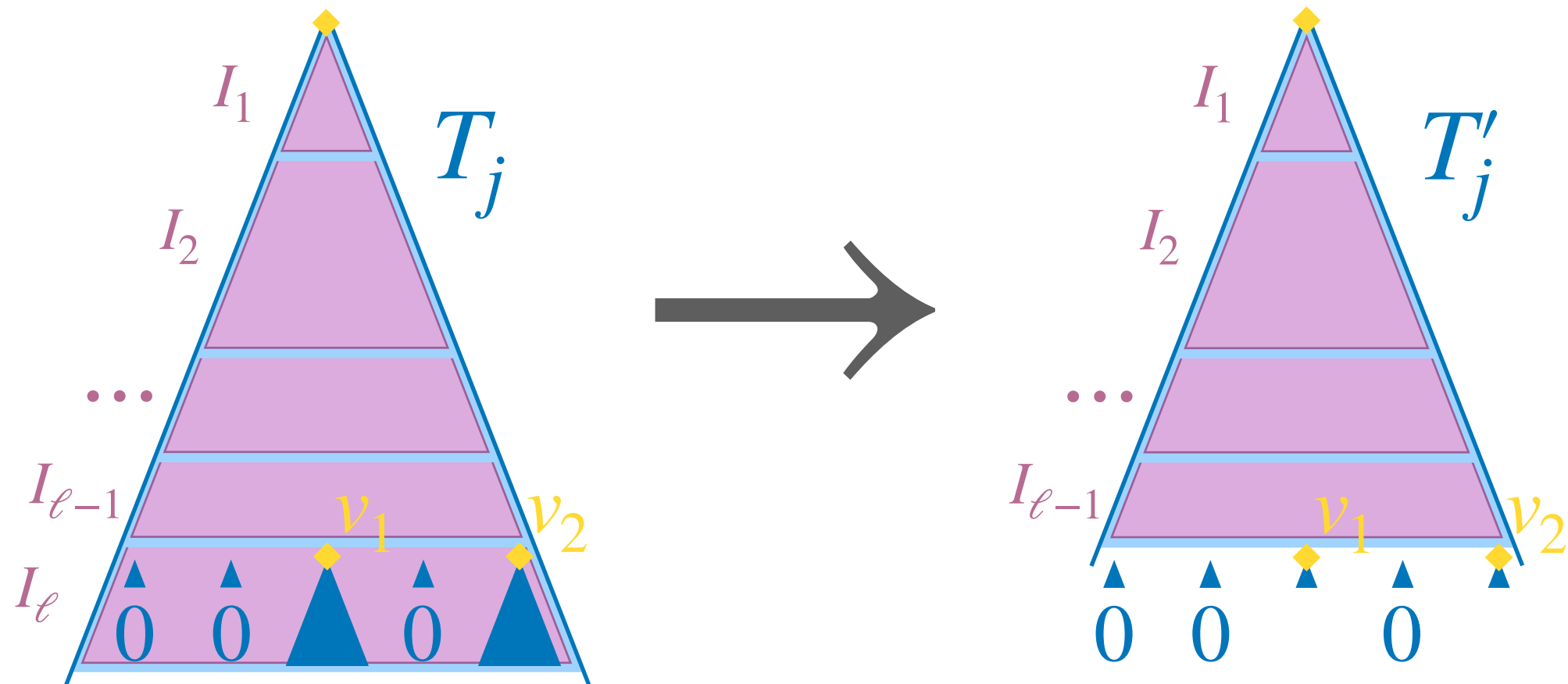
Analysis of the individual part

2. Replacing subtrees by their linear parts



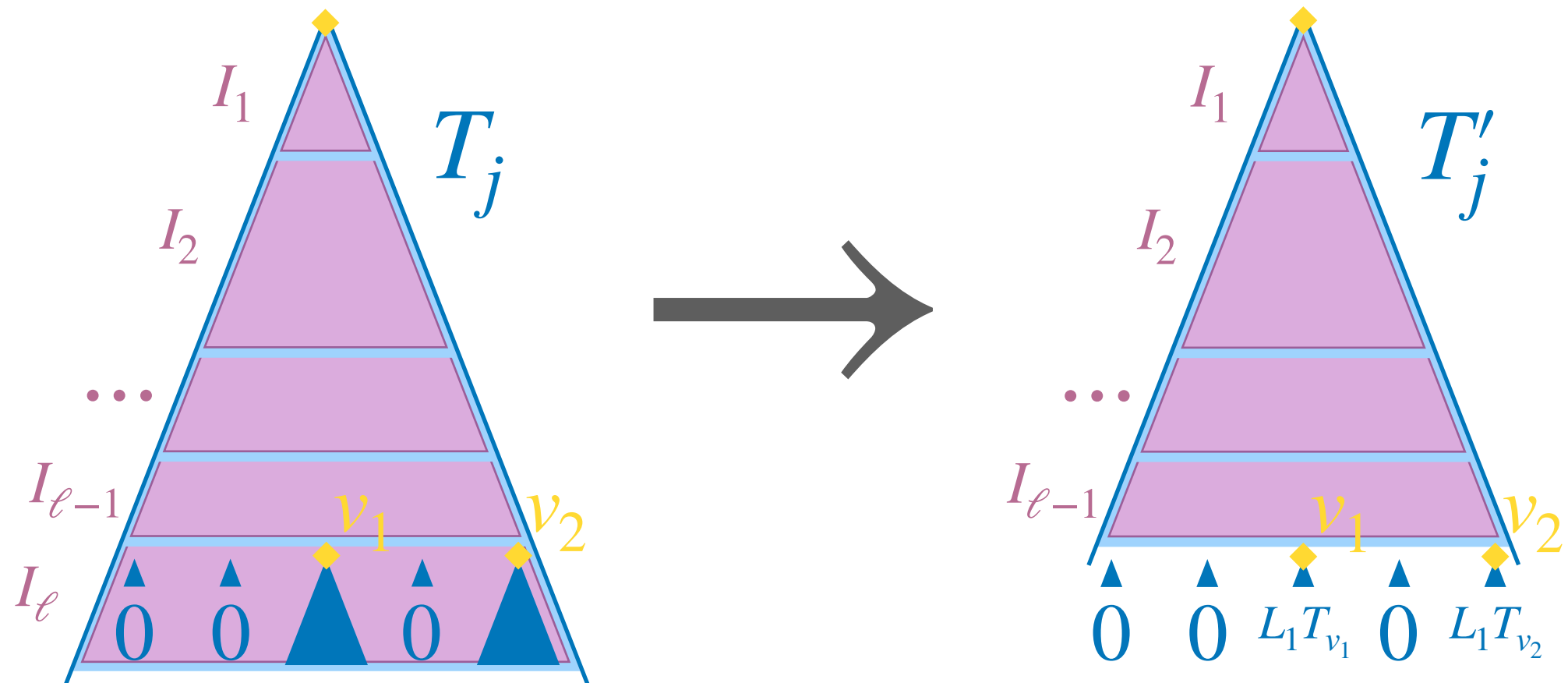
Analysis of the individual part

2. Replacing subtrees by their linear parts



Analysis of the individual part

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Analysis of the individual part

3. “Monomialization”

The diagram illustrates the process of monomialization, showing a triangular region T'_j on the left being equal to a sum over r of regions $T'_{j,r}$ on the right.

Left Triangle (T'_j):

- Vertices: A yellow diamond at the top, and two blue triangles at the base labeled $\sum a_i x_i$ (left) and $\sum b_i x_i$ (right).
- Edges: The left edge is divided into segments labeled $I_1, I_2, \dots, I_{\ell-1}$ from top to bottom. The right edge is labeled v_2 (yellow) and the bottom edge is labeled v_1 (yellow).

Right Triangle ($T'_{j,r}$):

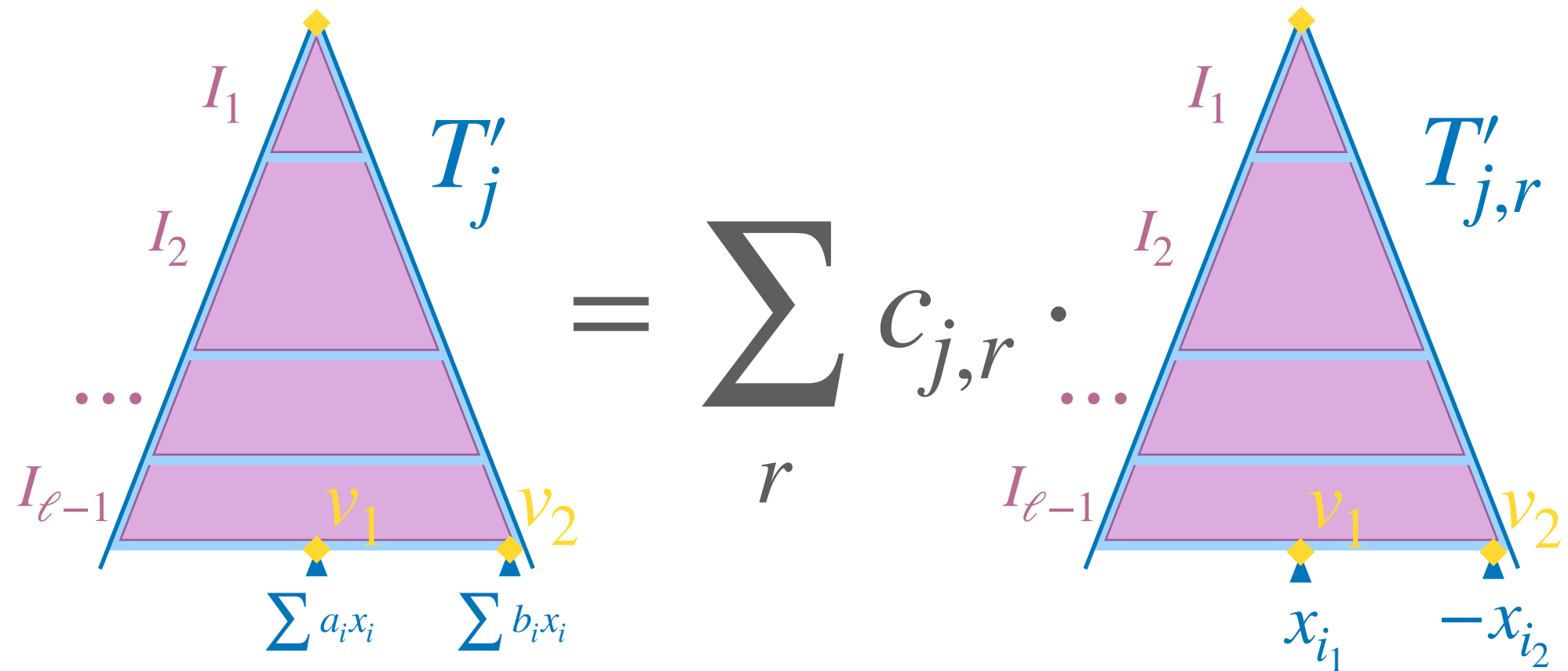
- Vertices: A yellow diamond at the top, and two blue triangles at the base labeled x_{i_1} (left) and $-x_{i_2}$ (right).
- Edges: The left edge is divided into segments labeled $I_1, I_2, \dots, I_{\ell-1}$ from top to bottom. The right edge is labeled v_2 (yellow) and the bottom edge is labeled v_1 (yellow).

The equation is represented as:

$$T'_j = \sum_r c_{j,r} \cdot T'_{j,r}$$

Analysis of the individual part

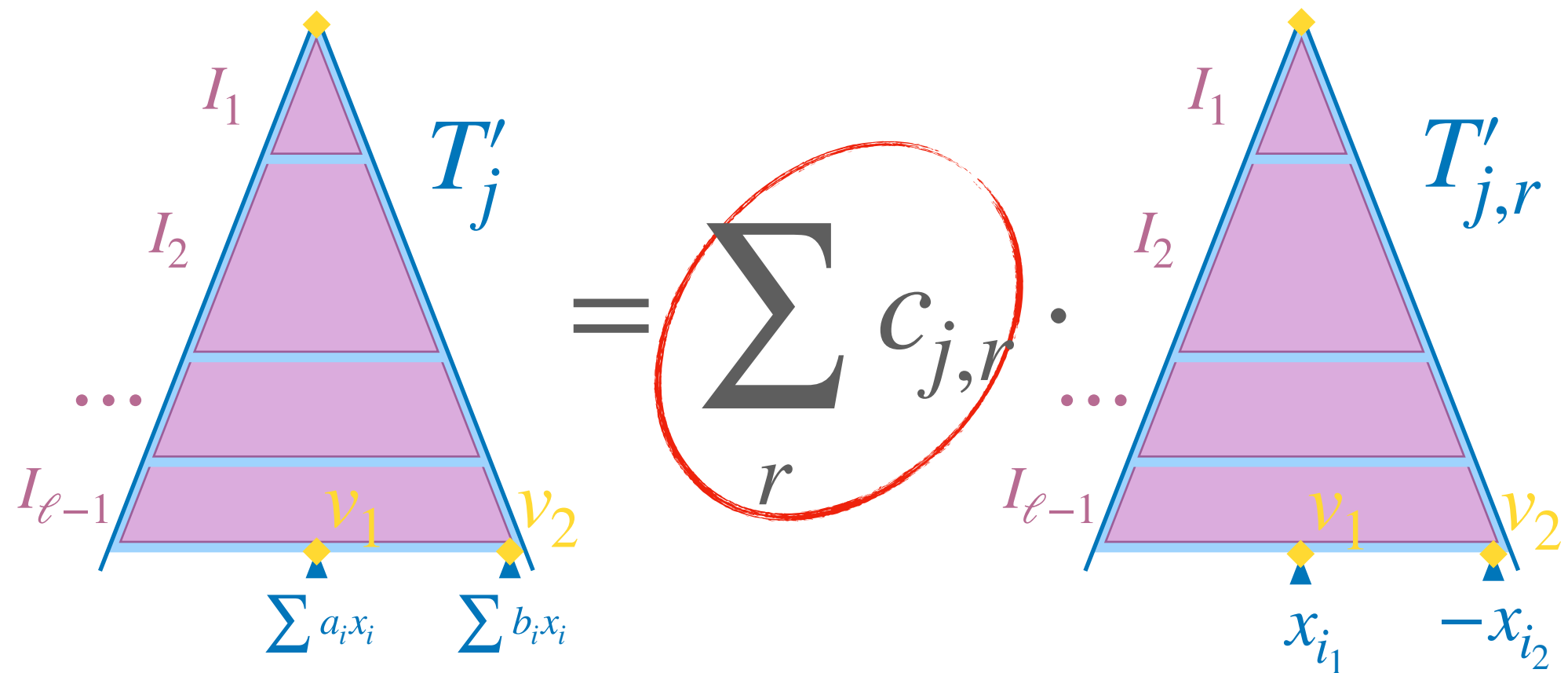
3. “Monomialization”



Make sure: $\text{dns}(T'_{j,r}) \leq \text{dns}(T'_j)$

Analysis of the individual part

3. “Monomialization”



Analysis of the individual part

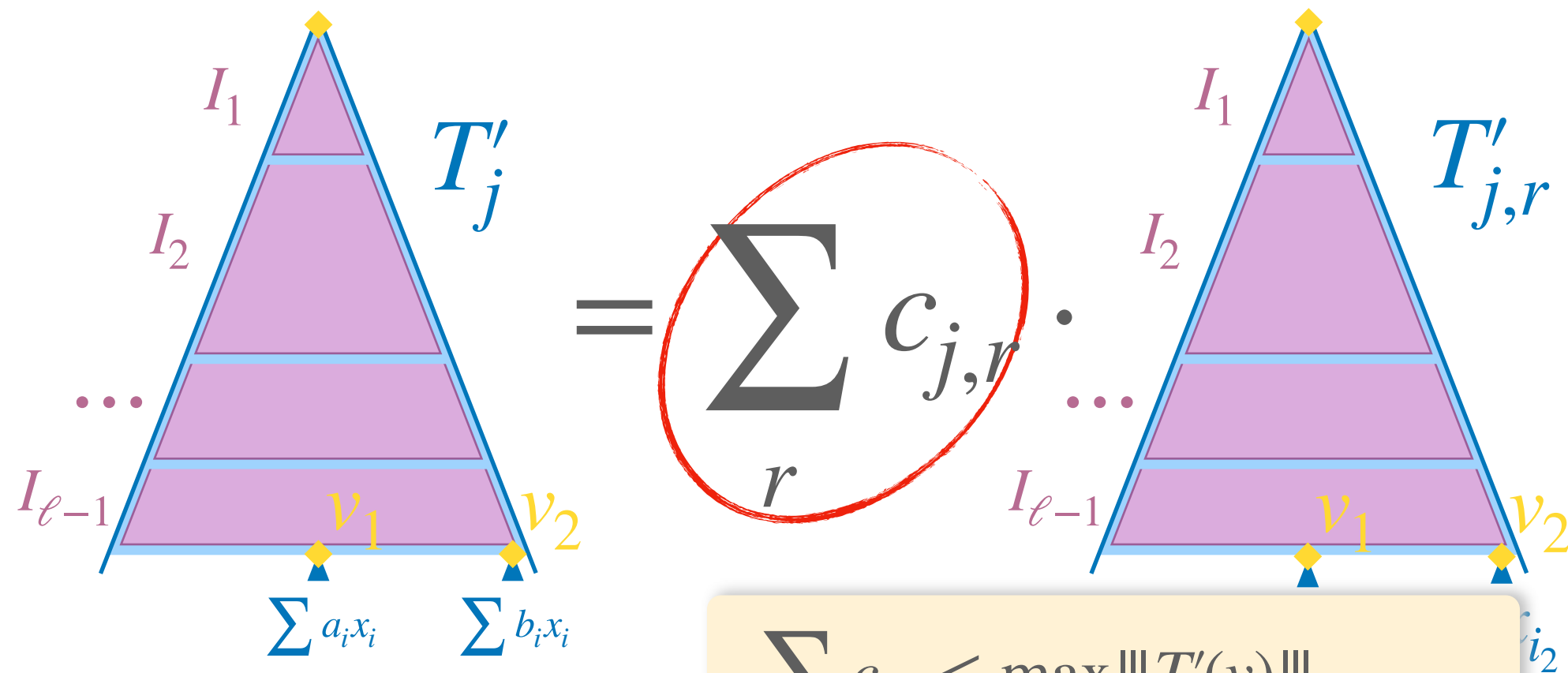
3. “Monomialization”

$$T'_j = \sum_r c_{j,r} \cdot T'_{j,r}$$

$$\sum_r c_{j,r} \leq \max_v ||| T'_j(v) |||$$

Analysis of the individual part

3. “Monomialization”

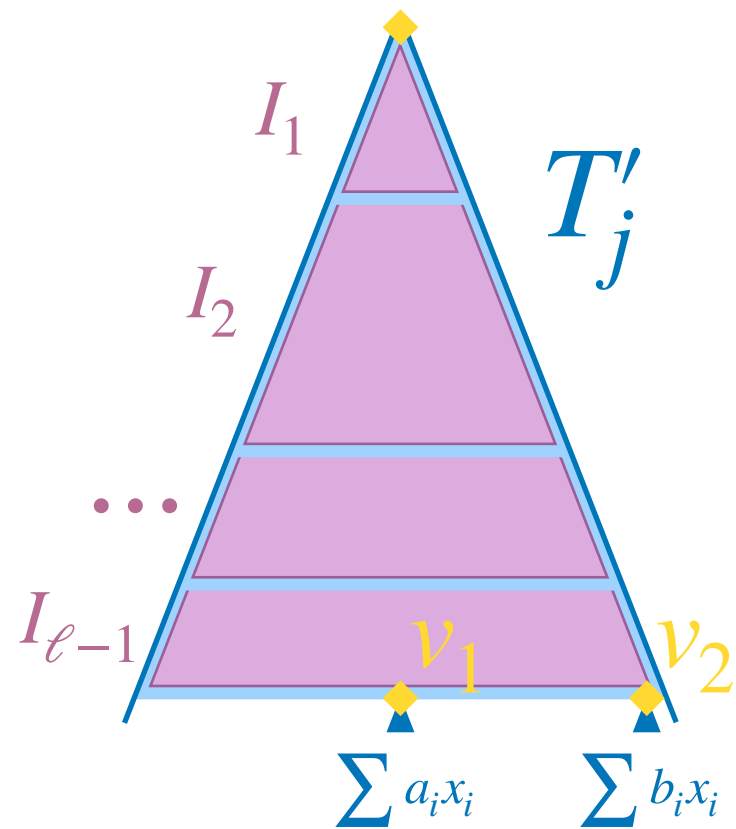


$$\sum_r c_{j,r} \leq \max_v ||| T'_j(v) |||$$

$$\leq c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j}).$$

Analysis of the individual part

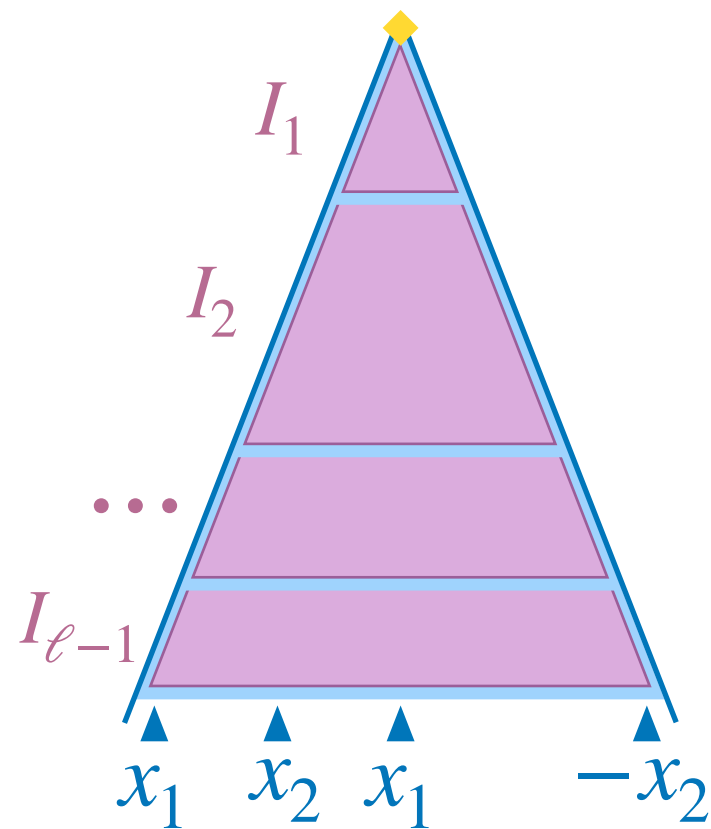
3. “Monomialization”



$$\in c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j}) \\ \times \text{conv} \{ \mathcal{T}_{\text{dns}(T_j)} \}$$

Analysis of the individual part

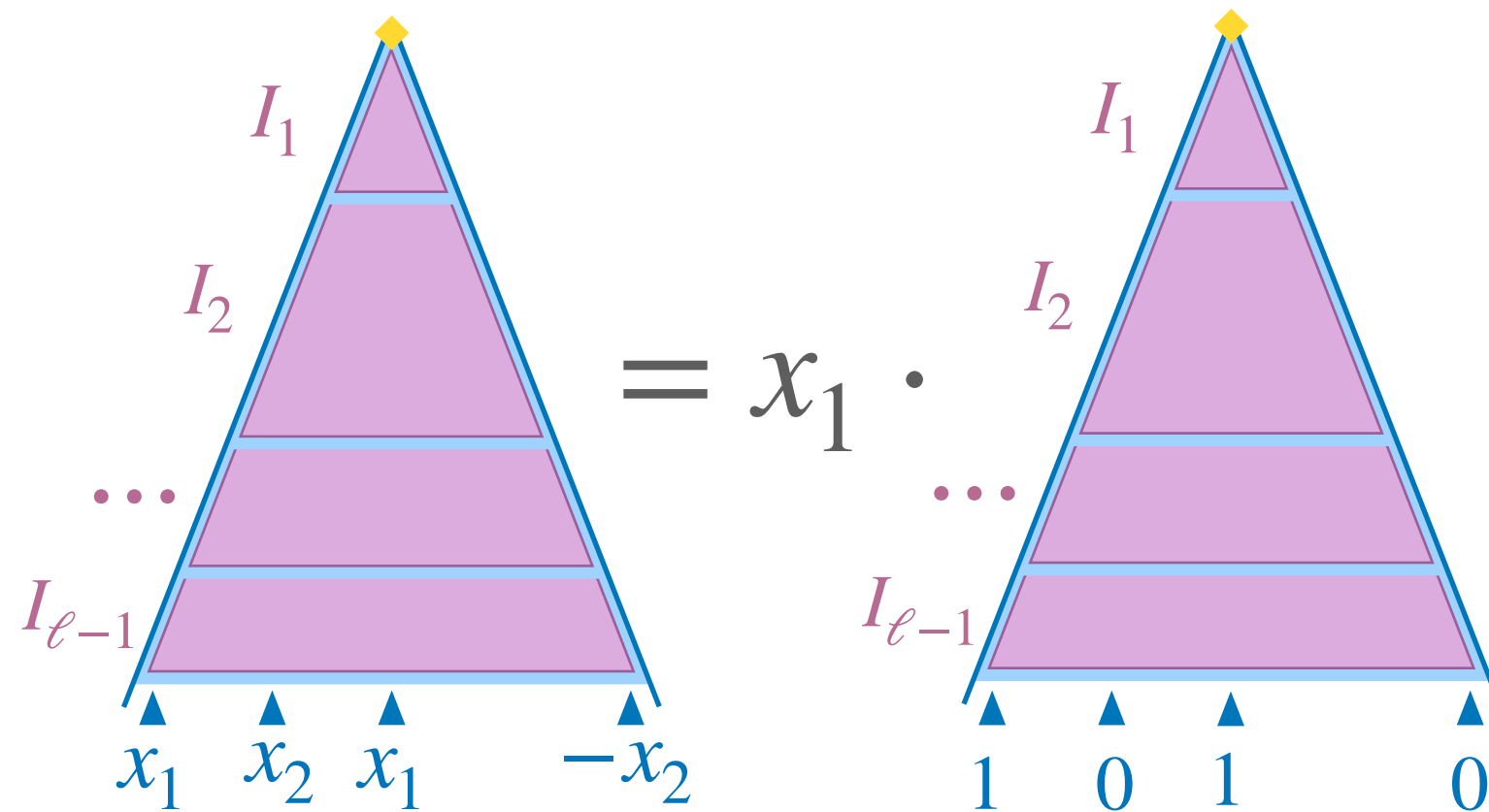
4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}}(T_j)$$

Analysis of the individual part

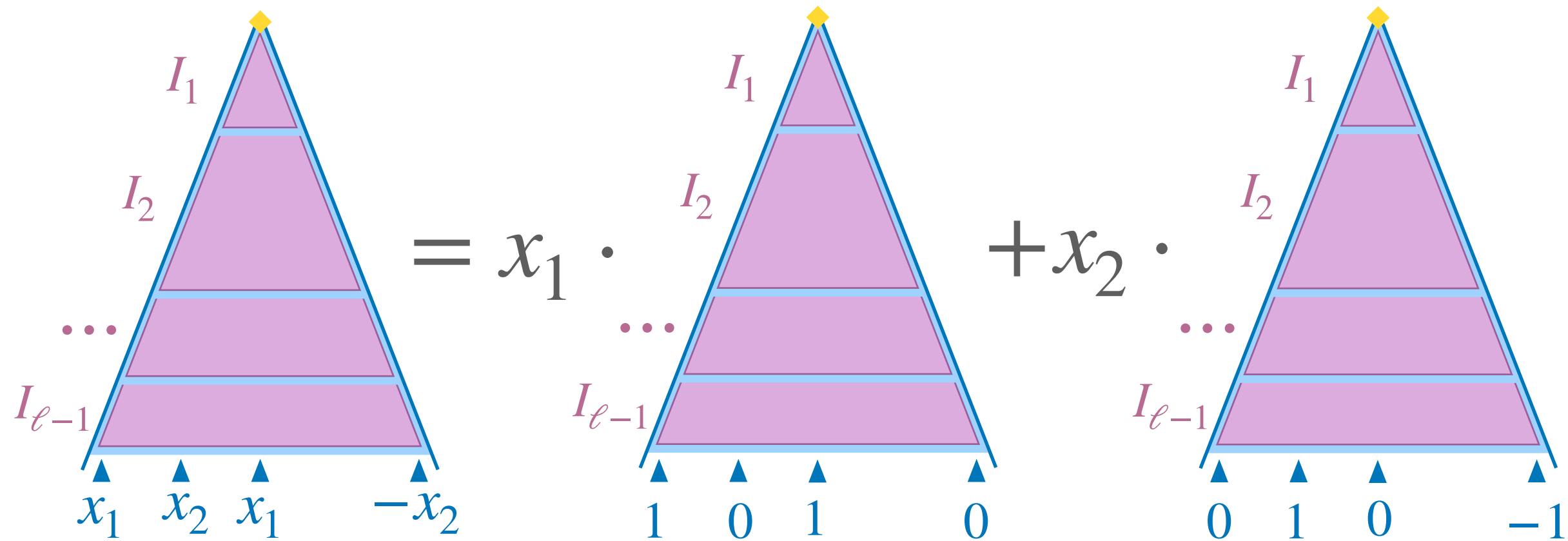
4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}}(T_j)$$

Analysis of the individual part

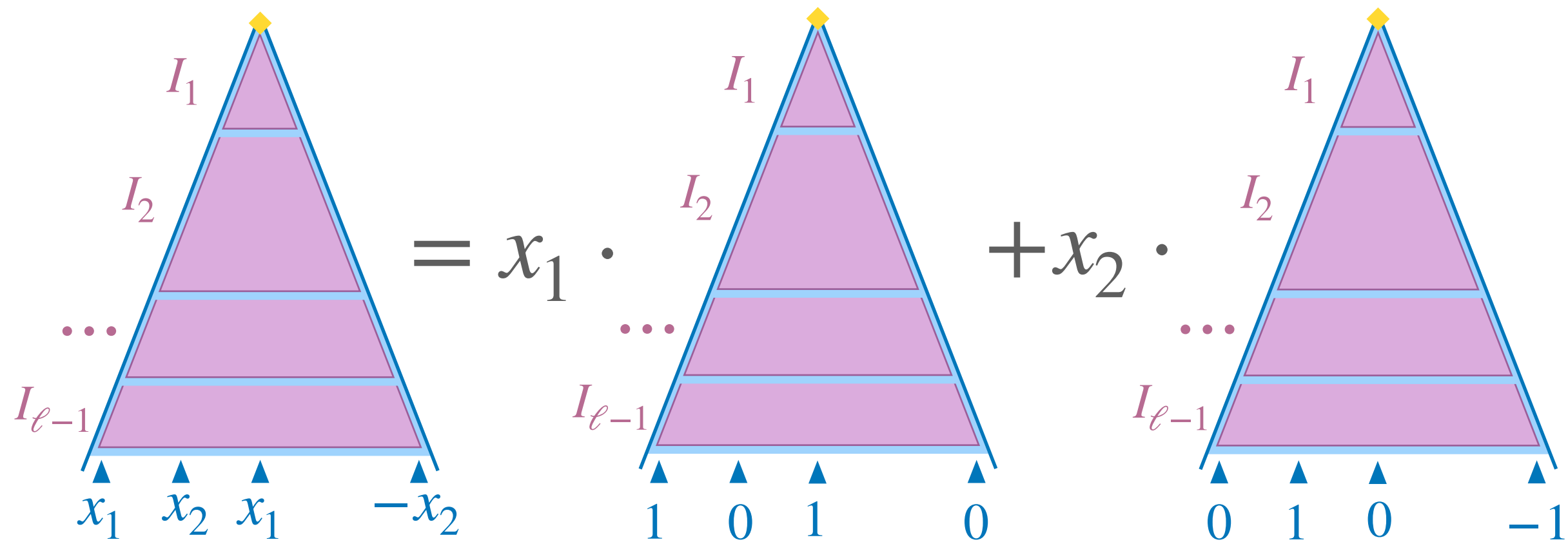
4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}(T_j)}$$

Analysis of the individual part

4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}(T_j)}$$

$+ \dots$

Analysis of the individual part

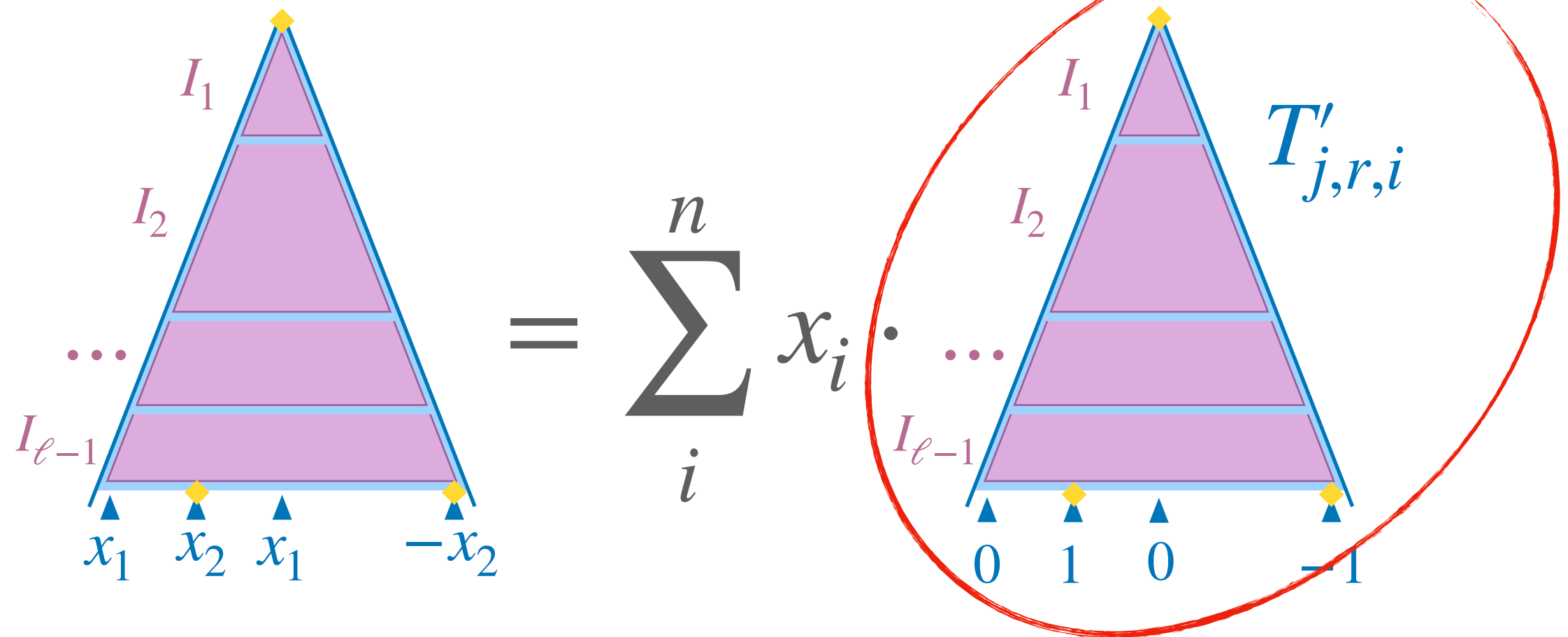
4. Grouping by variable

The diagram illustrates the decomposition of a triangle T_j into a sum of triangles $T'_{j,r,i}$ weighted by variables x_i . On the left, a triangle is divided into horizontal layers labeled $I_1, I_2, \dots, I_{\ell-1}$ from top to bottom. The bottom edge has four vertices marked with blue triangles and labeled $x_1, x_2, x_1, -x_2$ from left to right. A yellow diamond is at the top vertex. In the center is an equals sign followed by a summation $\sum_i^n x_i \cdot$. On the right, a similar triangle is shown, labeled $T'_{j,r,i}$, with the same horizontal layers $I_1, I_2, \dots, I_{\ell-1}$. Its bottom edge vertices are labeled $0, 1, 0, -1$ from left to right, and it also has a yellow diamond at the top vertex.

$$T'_{j,r} \in \mathcal{T}_{\text{dns}}(T_j)$$

Analysis of the individual part

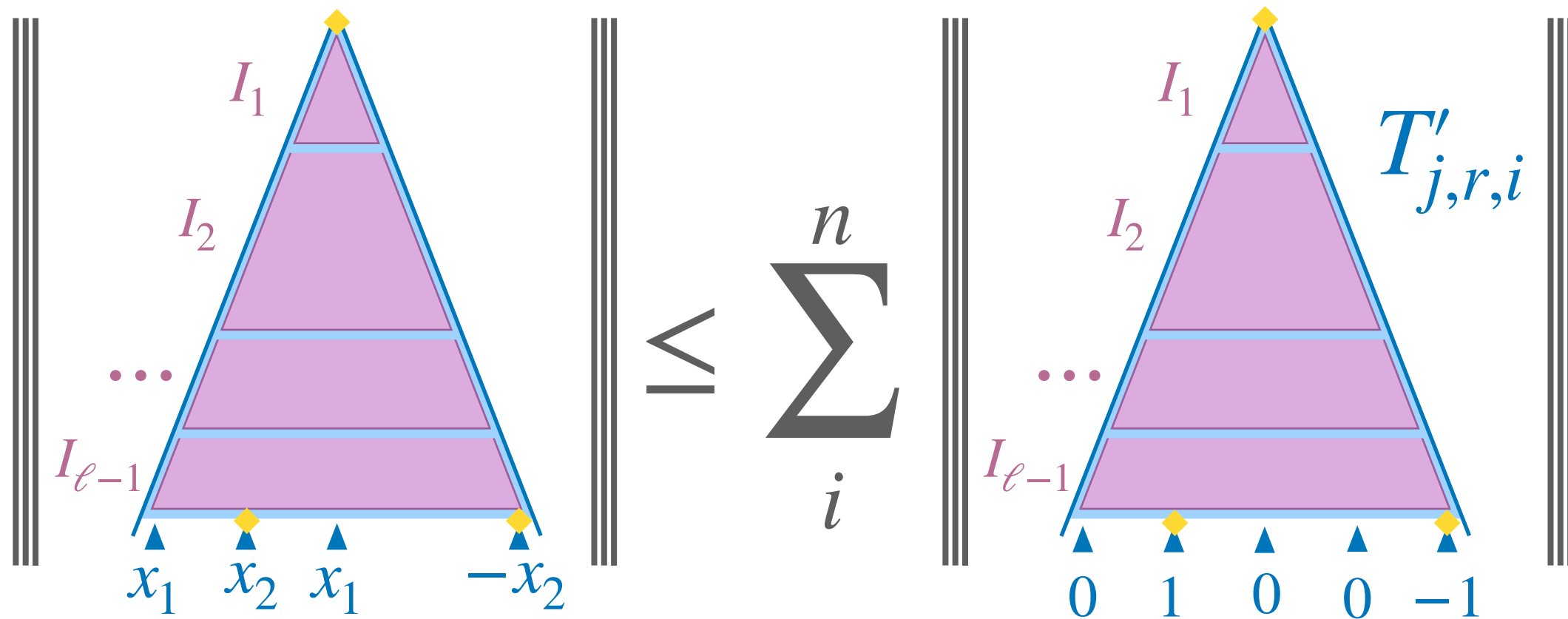
4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}}(T_j)$$

Analysis of the individual part

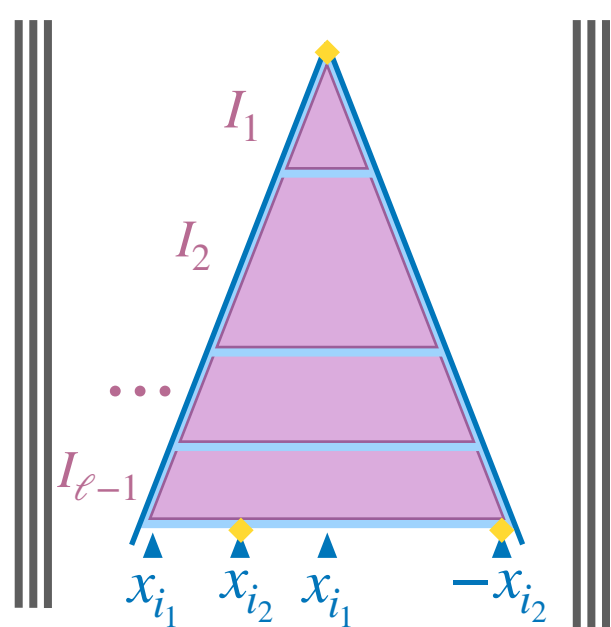
4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}(T_j)}$$

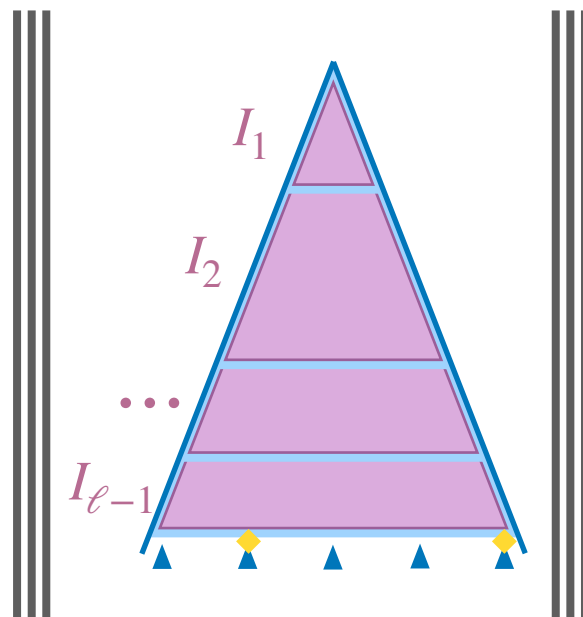
Analysis of the individual part

4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}(T_j)}$$

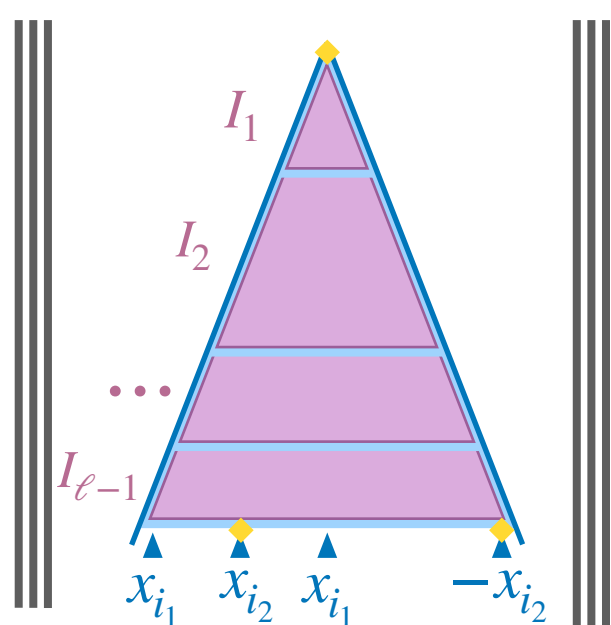
$$\leq \sum_i^n$$



$$\leq \sum_{i=1}^n c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s|} \Lambda_{n^2, \ell-1}(\text{dns}(T'_{j,r,i}))$$

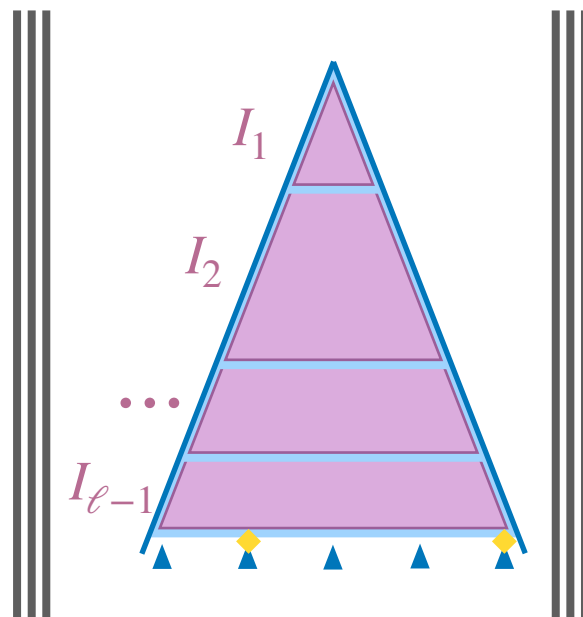
Analysis of the individual part

4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}(T_j)}$$

$$\leq \sum_i^n$$

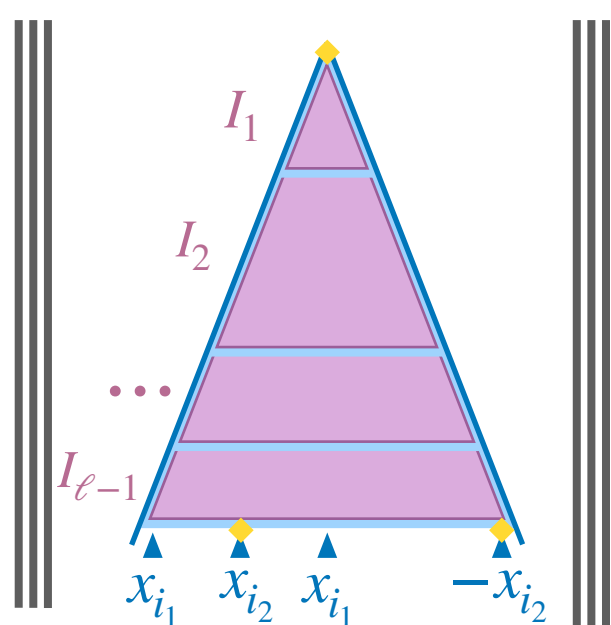


$$\leq \sum_{i=1}^n c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s|} \Lambda_{n^2, \ell-1}(\text{dns}(T'_{j,r,i}))$$

$$\stackrel{\text{(concavity)}}{\leq} c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s|} n \Lambda_{n^2, \ell-1} \left(\frac{\sum_{i=1}^n \text{dns}(T'_{j,r,i})}{n} \right)$$

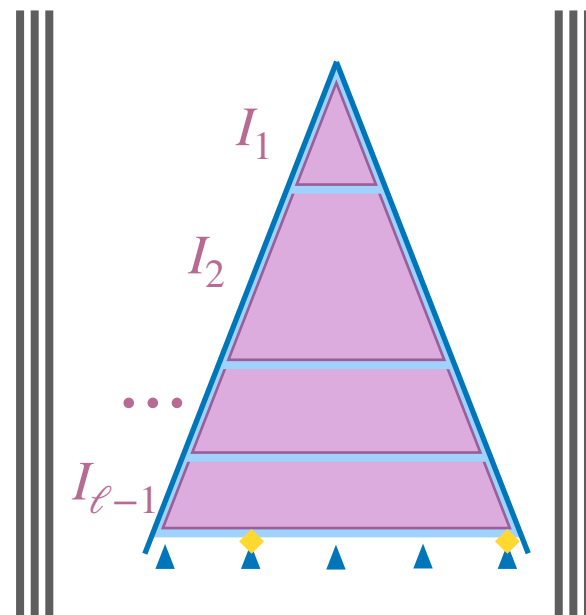
Analysis of the individual part

4. Grouping by variable



$$T'_{j,r} \in \mathcal{T}_{\text{dns}(T_j)}$$

$$\leq \sum_i^n$$



$$\leq \sum_{i=1}^n c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s| \Lambda_{n^2, \ell-1}(\text{dns}(T'_{j,r,i}))}$$

$$\stackrel{\text{(concavity)}}{\leq} c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s| n \Lambda_{n^2, \ell-1} \left(\frac{\text{dns}(T_j)}{n} \right)}$$

Analysis of the individual part

5. Putting things together

$$\begin{array}{c}
 \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ I_\ell \\ \text{▲} \text{▲} \text{▲} \text{▲} \text{▲} \\ T \end{array}
 \end{array}
 =
 \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n x_i \cdot
 \begin{array}{c}
 \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ \text{▲} \text{▲} \text{▲} \text{▲} \text{▲} \\ T'_{j,r,i} \end{array}
 \end{array}$$

$\longleftrightarrow$
 Bucketing

$\longleftrightarrow$
 Monomialization

$\longleftrightarrow$
 Group by variable

Analysis of the individual part

5. Putting things together

$$\begin{array}{c}
 \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ I_\ell \\ \text{▲} \text{▲} \text{▲} \text{▲} \text{▲} \\ T \end{array}
 \end{array}
 = \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n x_i \cdot \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ \text{▲} \text{▲} \text{▲} \text{▲} \text{▲} \\ T'_{j,r,i} \end{array}$$

$\longleftrightarrow$
 Bucketing

$\longleftrightarrow$
 $\in c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j})$
 $\times \text{conv}\{\mathcal{T}_{\text{dns}(T_j)}\}$

Analysis of the individual part

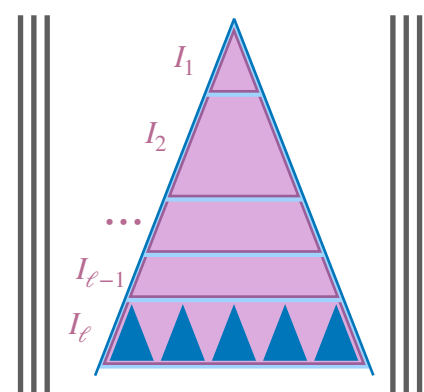
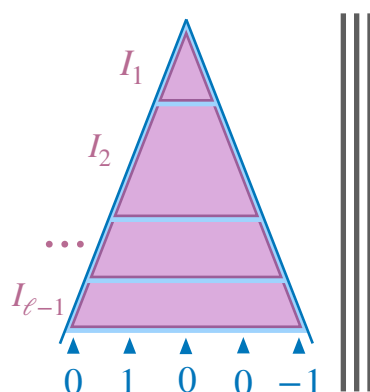
5. Putting things together

$$\begin{array}{c}
 \left(\begin{array}{c} \text{Triangle } T \text{ with levels } I_1, I_2, \dots, I_{\ell-1}, I_{\ell} \\ \text{Base of } T \text{ has } 5 \text{ blue triangles} \end{array} \right) \leq \sum_{j=0}^{\infty} \left(\sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n \left(\begin{array}{c} \text{Triangle } T'_{j,r,i} \text{ with levels } I_1, I_2, \dots, I_{\ell-1} \\ \text{Base of } T'_{j,r,i} \text{ has values } 0, 1, 0, 0, -1 \end{array} \right) \right) \\
 \xleftarrow{\text{Bucketing}} \\
 \xleftarrow{\in c_0 \sqrt{|I_{\ell}|} \Lambda_{n^2,1}(3^{-j}) \times \text{conv}\{\mathcal{T}_{\text{dns}(T_j)}\}}
 \end{array}$$

Analysis of the individual part

5. Putting things together

$$\begin{aligned}
 \left| \left| \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ I_\ell \\ \text{blue triangles} \end{array} \right| \right| &\leq \sum_{j=0}^{\infty} \left| \left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ \text{blue triangles} \end{array} \right| \right| \\
 &\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \text{conv}\{\mathcal{T}_{\text{dns}(T_j)}\}} \left| \left| T \right| \right|
 \end{aligned}$$

Analysis of the individual part

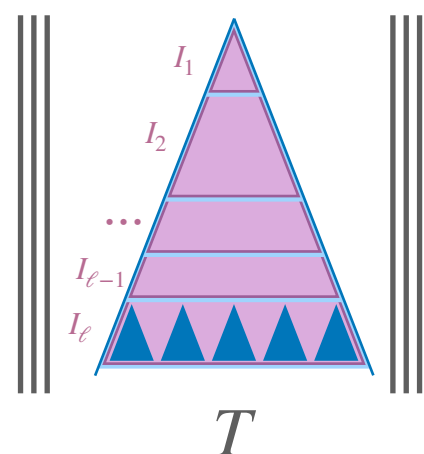
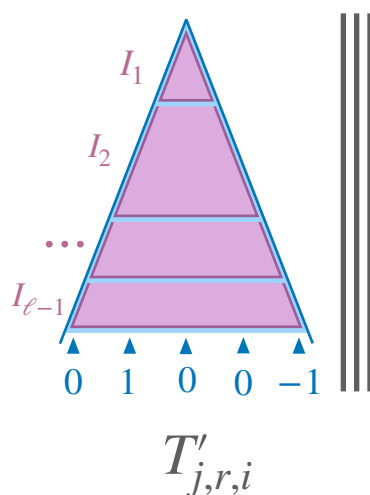
5. Putting things together

$$\begin{aligned}
 \left| \left| \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ I_\ell \end{array} \right. \right| \right| &\leq \sum_{j=0}^{\infty} \left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ \begin{array}{ccccc} 0 & 1 & 0 & 0 & -1 \end{array} \end{array} \right| \right| \\
 &\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \text{conv}\{\mathcal{T}_{\text{dns}(T_j)}\}} \left| \left| T \right| \right| \\
 &\leq c_0^{\ell-1} c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell-1} |I_s|} n \Lambda_{n^2,\ell-1} \left(\frac{\text{dns}(T_j)}{n} \right)
 \end{aligned}$$

Analysis of the individual part

5. Putting things together

$$\begin{aligned}
 \left| \left| \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ I_\ell \end{array} \right. \right| \right| &\leq \sum_{j=0}^{\infty} \left| \left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ \begin{array}{ccccc} 0 & 1 & 0 & 0 & -1 \end{array} \end{array} \right. \right| \right| \\
 &\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \text{conv}\{\mathcal{T}_{\text{dns}(T_j)}\}} \left| \left| T \right| \right| \\
 &\leq c_0^\ell c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell} |I_s|} \sum_{j=0}^{\infty} \Lambda_{n^2,1}(3^{-j}) n \Lambda_{n^2,\ell-1} \left(\frac{\text{dns}(T_j)}{n} \right)
 \end{aligned}$$

Analysis of the individual part

5. Putting things together

$$\begin{aligned}
 \left| \left| \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ I_\ell \\ \text{---} \end{array} \right. \right| \right| &\leq \sum_{j=0}^{\infty} \left| \left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ \text{---} \\ 0 \quad 1 \quad 0 \quad 0 \quad -1 \end{array} \right. \right| \right| \\
 &\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \text{conv}\{\mathcal{T}_{\text{dns}(T_j)}\}} \left| \left| T \right| \right| \\
 &\leq c_0^\ell c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell} |I_s|} \sum_{j=0}^{\infty} \Lambda_{n^2,1}(3^{-j}) n \Lambda_{n^2,\ell-1} \left(\frac{\text{dns}(T_j)}{n} \right) \\
 &\leq c_1 \Lambda_{n^2,\ell}(\text{dns}(T))
 \end{aligned}$$

Analysis of the individual part

5. Putting things together

$$\begin{aligned}
 \left| \left| \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ I_\ell \end{array} \right. \right| \right| &\leq \sum_{j=0}^{\infty} \left| \sum_{r=1}^{\infty} c_{j,r} \sum_{i=1}^n \begin{array}{c} I_1 \\ I_2 \\ \dots \\ I_{\ell-1} \\ \begin{array}{ccccc} 0 & 1 & 0 & 0 & -1 \end{array} \end{array} \right| \right| \\
 &\leq \sum_{j=0}^{\infty} c_0 \sqrt{|I_\ell|} \Lambda_{n^2,1}(3^{-j}) \cdot \max_{T \in \text{conv}\{\mathcal{T}_{\text{dns}(T_j)}\}} \left| \left| T \right| \right| \\
 &\leq c_0^\ell c_1^{\ell-2} \sqrt{\prod_{s=1}^{\ell} |I_s|} \sum_{j=0}^{\infty} \Lambda_{n^2,1}(3^{-j}) n \Lambda_{n^2,\ell-1} \left(\frac{\text{dns}(T_j)}{n} \right) \\
 &\leq c_1 \Lambda_{n^2,\ell}(\text{dns}(T))
 \end{aligned}$$

Open problems

Problem 1

For total functions f , does one have $R_{1/3}(f) \leq O(Q_{1/3}(f)^3)$?

Problem 2

Are randomized and quantum communication complexity polynomially related for total functions?

Open problems

Problem 1

For total functions f , does one have $R_{1/3}(f) \leq O(Q_{1/3}(f)^3)$?

Problem 2

Are randomized and quantum communication complexity polynomially related for total functions?

Thank you!